Three topics on Hilbert functions:

Fröberg's Conjecture, Hilbert coefficients and deformations of local rings

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To Professor Maria Evelina Rossi and my family

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Contents

1. Introduction

The notion of Hilbert Function is central in Commutative Algebra and it has important applications in Algebraic Geometry, Combinatorics, Singularity Theory and Computational Algebra.

In this thesis we shall deal with different aspects of the theory of Hilbert functions by presenting our contribution concerning three problems of great interest through the last decades in this area of dynamic mathematical activity.

The first part of the thesis concerns the study of the Hilbert function of standard graded algebras. In his famous paper "Über die Theorie der algebraischen Formen" (see [28]) published more than a century ago, Hilbert proved that a graded module M over a polynomial ring has a finite graded resolution and he concluded from this fact that its Hilbert function is of polynomial type. The Hilbert function of the homogeneous coordinate ring of a projective variety V, which classically was called the postulation of V, is a rich source of discrete invariants of V and its embedding. The dimension, the degree and the arithmetic genus of V can be immediately read from the generating function of the Hilbert function.

Recall that if $A = K[x_1, \cdots, x_n]/I$, where I is an homogeneous ideal of the polynomial ring $R = K[x_1, \cdots, x_n]$, then the Hilbert function of the standard graded algebra A is by definition

$$H_A(t) := \dim_K R_t/I_t \text{ for every } t \geq 0,$$

where $(\ )_t$ denotes the homogeneous part of degree t, and the Hilbert series of A is by definition

$$P_A(z) := \sum_{t\geq 0} H_A(t) z^t.$$

The possible Hilbert functions of a homogeneous Cohen-Macaulay algebra are easily characterized by using Macaulay's theorem and a standard graded prime avoidance theorem. Hence the original problems can be reduced to the study of the Hilbert function of an Artinian graded algebra. One of the delightful things of our subject is that one can begin studying it in an elementary way and, all of a sudden, one can front extremely challenging and interesting problems, even in the Artinian case. Our attention is devoted to a longstanding problem on the Hilbert function of generic graded algebras, a conjecture stated by R. Fröberg in 1985, see [18].

Let $R = K[x_1, \cdots, x_n]$ be the polynomial ring over a field K and $A = R/I$ a graded standard algebra, where I is an homogeneous ideal of R. If $F \in R_j$ is a generic form, it is very natural to guess that for every $t \geq 0$, the multiplication map

$$A_t \xrightarrow{.F} A_{t+j}$$

is of maximal rank, which means that it is injective if $\dim_K A_t \leq \dim_K A_{t+j}$, and surjective if $\dim_K A_t \geq \dim_K A_{t+j}$. Since for every $F \in R_j$ we have an exact sequence

$$0 \to (0 :_A F)(-j) \to A(-j) \xrightarrow{.F} A \to A/FA \to 0.$$

We get

$$H_{A/FA}(t) = H_A(t) - H_A(t-j) + H_{0:F}(t-j).$$

Hence if F is generic (see Definition 2.2.1 for the definition) we guess the following equality

$$H_{A/FA}(t) = \max\{0, H_A(t) - H_A(t-j)\},$$

or in terms of the Hilbert series

$$P_{A/FA}(z) = \lceil (1 - z^j) P_A(z) \rceil .$$

where for a power series $\sum a_i z^i$ we denote $\lceil \sum a_i z^i \rceil := \sum b_i z^i$, with $b_i = a_i$ if $a_j > 0$ for all $j \leq i$, and $b_i = 0$ otherwise. In 1985, Fröberg stated the following conjecture: Let $F_1, \dots, F_r$ be generic forms in R of degree $d_1, \dots, d_r$ and we will say that the ideal generated is of type $(n; d_1, \cdots, d_r)$. If $I = (F_1, \dots, F_r)$, then

$$P_{R/I}(z) = \left\lceil \frac{\prod_{i=1}^{r} (1 - z^{d_i})}{(1 - z)^n} \right\rceil .$$

This problem is of central interest in commutative algebra in the last decades and a great deal was done (see for instance Anick [3], Fröberg [18], Fröberg-Hollman [19], Fröberg-Löfwall [20], Fröberg-Lundqvist [21], Moreno-Socías [33], Pardue [40], Stanley [60], Valla [67]). A large number of validations through computational methods suggests a positive answer. Fröberg's Conjecture is clearly true if $r \leq n$ (complete intersections); it is known if $n \leq 2$ [18]; $n = 3$ [3]; $r = n + 1$ (almost complete intersection) with $\operatorname{char} K = 0$ [61]; and some further special cases when all d_i are equal (see [6], [19], [38]). Our study will contribute to give a new partial solution to Fröberg's Conjecture in the cases $r = n + 1, n + 2$ for any characteristic under a suitable condition on $d_1, \cdots, d_r$ (see Theorem 2.7.6). Actually our main goal is to prove an equivalent conjecture stated by Pardue which will be presented below.

Denote by $\operatorname{in}_\tau(I)$ the initial ideal of I with respect to a term order τ on R. Because the Hilbert series of R/I and of $R/\operatorname{in}_\tau(I)$ coincide for every τ, a rich literature has been developed with the aim to characterize the initial ideal of generic ideals with respect to suitable term orders (see [1], [3], [11], [10], [12], [30], [32], [33], [40]). From now on, the initial ideal of I will be always with respect to the *degree reverse lexicographic order* and it will be denoted simply by $\operatorname{in}(I)$. It is natural to guess that generic complete intersections share *special* initial ideals.

Pardue stated a conjecture on the initial ideal of a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $R = K[x_1, \cdots, x_n]$ which is equivalent to Fröberg's Conjecture ([40, Theorem 2]).

Actually in [32], Moreno-Socías stated a stronger conjecture announcing that $\operatorname{in}(I)$ should be almost reverse lexicographic, i.e, if x^μ is a minimal generator of $\operatorname{in}(I)$ then every monomial of the same degree and greater than x^μ must be in $\operatorname{in}(I)$ as well. Moreno-Socías' Conjecture was proven in the case $n = 2$ by Aguire et al. [1] and Moreno-Socías [33], $n = 3$ by Cimpoeas [12], $n = 4$ by Harima and Wachi [30] and for certain sequences $d_1, \cdots, d_n$ by Cho and Park assuming $\operatorname{char} K = 0$ [10]. Without restriction on the characteristic of K, by using an incremental method introduced in [23], Capaverde and Gao improved the result of Cho and Park, see [11, Theorem 3.19].

Inspired by the incremental method by Capaverde and Gao, in Proposition 2.6.8, we give an explicit description of the initial ideal of generic ideals with respect to the degree reverse lexicographic order. From this description, we obtained a partial solution to Pardue's Conjecture under suitable conditions on the degree of the generic forms (see Theorem 2.7.3 and Theorem 2.7.4). We hope that this approach will be successfully applied to give new insights in proving Pardue's Conjecture and hence Fröberg's Conjecture.

Let $d_1 \leq \cdots \leq d_r$ and for every $1 \leq i \leq r$, we set

$$\delta_i = d_1 + \cdots + d_i - i,$$

$$\sigma_i = \min\left\{\delta_{i-1}, \left\lfloor\frac{\delta_i}{2}\right\rfloor\right\} \text{ for all } i \geq 2.$$

Our main contribution to Fröberg's Conjecture is the following.

Theorem 1.0.1. (see Theorem 2.7.6.) *Let $I = (f_1, \cdots, f_r)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_r)$ in $R = K[x_1, \cdots, x_n]$ with $r \leq n+2$ and $d_1 \leq \cdots \leq d_r$. If $r \leq 3$ or $r \geq 4$ and $d_i \geq \sigma_{i-1}$ for every $4 \leq i \leq r$, then the Hilbert series of R/I is given by*

$$HS_{R/I}(z) = \left\lceil \frac{\prod_{i=1}^{r}(1-z^{d_i})}{(1-z)^n} \right\rceil.$$

Let now $(R, \mathfrak{m}, K)$ a Noetherian commutative local ring with maximal ideal $\mathfrak{m}$ and residue field K. The Hilbert function of R is, by definition, the numerical function

$$H_R(t) := \dim_K(\mathfrak{m}^t/\mathfrak{m}^{t+1}),$$

hence it coincides with the above defined Hilbert function of the standard graded algebra $gr_{\mathfrak{m}}(R) := \bigoplus_{t\geq 0}(\mathfrak{m}^t/\mathfrak{m}^{t+1})$, which is called the associated graded ring or tangent cone of R. The Hilbert function of a local ring $(R, \mathfrak{m})$ is a classical invariant which give information on the corresponding singularity. The reason is that the associated graded algebra $gr_{\mathfrak{m}}(R)$ corresponds to an important geometry construction: namely, if R is the localization at the origin of the coordinate ring of an affine variety V passing through 0, then $gr_{\mathfrak{m}}(R)$ is the coordinate ring of the cone composed of all lines that are limiting positions of secant lines to V in 0.

Despite the fact that the Hilbert function of a standard graded algebra is well understood in the case R is Cohen-Macaulay, very little is known in the local case because passing to $gr_{\mathfrak{m}}(R)$ we may lose many good properties of the base ring R. Due to the pioneering work made by D.G. Northcott in 50's (see for instance [36] and [37]), or by J. Sally in 1990 (see for instance [52]-[59]), several efforts have been made to better understand Hilbert function of a local ring, also in relation with its Hilbert coefficients which give an asymptotic information. A great interest is the extension to modules and to the filtrations of modules. A large literature has been produced on this topic, see for instance the results obtained by D. Northcott, J. Fillmore, C. Rhodes, D. Kirby, H. Meheran and, more recently, T. Cortadellas and S. Zarzuela, J. Verma, T. Puthenpurakal, M.E. Rossi and J. Valla, V. Trivedi who carried over to the general setting.

We remark that the graded algebra $gr_{\mathfrak{m}}(R)$ can also be seen as the graded algebra associated to an ideal filtration of the ring itself, namely the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^j\}_{j\geq 0}$. This gives an indication of a possible natural extension of the theory to general filtrations of a finite module over the local ring $(R, \mathfrak{m})$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R, we consider $\mathbb{M} = \{M_n\}$ a $\mathfrak{q}$-filtration of M as follows

$$\mathbb{M}: \quad M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

where M_n are submodules of M and $\mathfrak{q}M_n \subseteq M_{n+1}$ for all $n \geq 0$. The $\mathfrak{q}$-filtration $\mathbb{M}$ is called a good $\mathfrak{q}$-filtration if $\mathfrak{q}M_n = M_{n+1}$ for all sufficiently large n.

The algebraic and geometric properties of M can be detected by the Hilbert function of a good $\mathfrak{q}$-filtration $\mathbb{M}$, namely $H_{\mathbb{M}}(n) = \lambda(M_n/M_{n+1})$, where $\lambda(-)$ denotes the length as R-module. It is well known that there exist the integers $e_i(\mathbb{M})$ for $i = 0, 1, \cdots, d$ such that for $n \gg 0$

$$H_{\mathbb{M}}(n) = e_0(\mathbb{M})\binom{n+d-1}{d-1} - e_1(\mathbb{M})\binom{n+d-2}{d-2} + \cdots + (-1)^{d-1}e_{d-1}(\mathbb{M}).$$

The integers $e_i(\mathbb{M})$ are called the Hilbert coefficients of $\mathbb{M}$. In particular $e = e_0(\mathbb{M})$ is the multiplicity and it depends only on M and $\mathfrak{q}$.

A rich literature has been produced on the Hilbert coefficients of a filtered module M in the case M is Cohen-Macaulay, for a survey see for instance [45]. The first Hilbert coefficient $e_1(\mathbb{M})$ is called Chern number by W.V. Vasconcelos and has been studied very deeply by several authors (see for instance [14], [25], [29], [36], [40] and [46]).

Let J be an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. If M is Cohen-Macaulay then the following inequalities hold

$$0 \le e_0(\mathbb{M}) - \lambda(M/M_1) \le e_1(\mathbb{M}) \le \sum_{n\ge 0}\lambda(M_{n+1}/JM_n)$$

(see [25] and [29]) and the equalities provide good homological properties of the associated graded module $gr_{\mathbb{M}}(M) = \bigoplus_{n\ge 0} M_n/M_{n+1}$. Concerning the second Hilbert coefficient $e_2(\mathbb{M})$, if M is Cohen-Macaulay, then

$$0 \le e_2(\mathbb{M}) \le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n) \qquad (*)$$

(see [29] and [45]). If M is no longer Cohen-Macaulay then new tools are necessary. The first Hilbert coefficient $e_1(\mathbb{M})$ was studied by Goto-Nishida in [24] and by Rossi-Valla in [46]. We have the following inequalities

$$e_0(\mathbb{M}) - \lambda(M/M_1) \le e_1(\mathbb{M}) - e_1(\mathbb{N}) \le \sum_{n\ge 0}\lambda(M_{n+1}/JM_n)$$

where $\mathbb{N} = \{J^nM\}$ is the J-adic filtration on M.

Little is known about $e_2(\mathbb{M})$. In [34] Mccune proved that if $\operatorname{depth} R \ge d-1$ and $\mathfrak{q}$ is a parameter ideal of R, then the second Hilbert coefficient of the $\mathfrak{q}$-adic filtration $\{\mathfrak{q}^n\}$ on R is non-positive. In this thesis we extend the inequalities $(*)$ to the case M has almost maximal depth and we recover Mccune's result. More precisely, we prove the following main results.

Theorem 1.0.2. (see Theorem 3.6.1.) *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension $d \ge 2$ and* $\operatorname{depth} M \ge d-1$. *Suppose $J = (a_1, \cdots, a_d)$ is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. For each $i = 1, \cdots, d-1$, denote the ideal $J_i = (a_1, \cdots, a_{d-i})$ of R. Then, we have*

$$e_2(\mathbb{M}) \le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n).$$

Further, the equality holds if and only if $\operatorname{depth} gr_{\mathbb{M}}(M) \ge d-1$ *and* $(J_1M :_M a_d) \cap M_1 = J_1M$.

Example 3.6.3 shows that the assumption on the depth of M can not be weakened in the above result. Further if we consider $M = R$ and $\mathbb{N} = \{J^n\}$ is the J-adic filtration of R, then our result implies the non-positivity of $e_2(\mathbb{N})$ which was proved by Mccune as above mentioned.

If M is Cohen-Macaulay, then $e_2(\mathbb{N}) = 0$ and $gr_{\mathbb{N}}(M)$ is Cohen-Macaulay too. Under the assumption that the $gr_{\mathbb{N}}(M)$ has almost maximal depth, we may strengthen Theorem 1.0.2.

Theorem 1.0.3. (see Theorem 3.6.6.) *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension $d \geq 2$. Suppose J is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$ such that* $\operatorname{depth} gr_{\mathbb{N}}(M) \geq d-1$*, where $\mathbb{N}$ is the J-adic filtration. Then, we have*

$$e_2(\mathbb{M}) - e_2(\mathbb{N}) \leq \sum_{n \geq 1} n\lambda(M_{n+1}/JM_n).$$

A lower bound for $e_2(\mathbb{M})$ is also given and it extends the result by Rees-Narita on the non-negativity of the second Hilbert coefficient proved in the Cohen-Macaulay case.

Theorem 1.0.4. (see Theorem 3.5.7.) *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension two and* $\operatorname{depth} M > 0$*. Suppose $J = (a_1, a_2)$ is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then, we have*

$$e_2(\mathbb{M}) \geq -\binom{s+2}{2}\lambda\left(\frac{a_1M : a_2}{a_1M}\right),$$

where s is the postulation number of the Ratliff-Rush filtration associated to $\mathbb{M}$.

In the last part of the thesis we present a third problem of a great interest in Commutative algebra: the study of classical invariants of a local ring under small perturbation. The results concerning this part are obtained jointly with P.H. Quy.

This part is inspired by the recent work of L. Ma, P. H. Quy and I. Smirnov [35] about the preservation of Hilbert function under sufficiently small perturbations which was inspired by the previous work of Srinivas and Trivedi [62]. Taking a small perturbation arises naturally in studying deformations when we change the defining equations by adding terms of high order. In this way we can transform a singularity defined analytically, e.g., as a quotient of a (convergent) power series ring, into an algebraic singularity by truncating the defining equations.

This problem was first considered by Samuel in 1956. Let $f \in S = k[[x_1, \ldots, x_d]]$ be a hypersurface with an isolated singularity, i.e. the Jacobian ideal $J(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d})$ is $(x_1, \ldots, x_d)$-primary. Then Samuel proved that for every $\varepsilon \in (x_1, \ldots, x_d)J(f)^2$ we have an automorphism of S that maps $f \mapsto f + \varepsilon$. In particular, Samuel's result asserts if f has an isolated singularity and ε is in a sufficiently large power of $(x_1, \ldots, x_d)$, then the rings $S/(f)$ and $S/(f+\varepsilon)$ are isomorphic. Samuel's result was extended by Hironaka in 1965, who showed that if S/I is an equidimensional reduced isolated singularity, then $S/I \cong S/I'$ for every ideal I' obtained by changing the generators of I by elements of sufficiently large order such that S/I' is still reduced, equidimensional, and same height as I.

The isolated singularity is essential in both Samuel and Hironaka's theorem. For a local ring $(R, \mathfrak{m})$ and a sequence of elements $\underline{f} = f_1, \ldots, f_r$, instead of requiring the deformation to give isomorphic rings $R/(f_1, \ldots, f_r) \cong R/(f_1+\varepsilon_1, \ldots, f_r+\varepsilon_r)$, we consider a weaker question: what properties and invariants are preserved by a sufficiently fine perturbation? For example, Eisenbud [15] showed how to control the homology of a complex under a perturbation and thus showed that Euler characteristic and depth can be preserved. As an application, if $\underline{f} = f_1, \ldots, f_r$ is a regular sequence, then so is the sequence $\underline{f_\varepsilon} = f_1 + \varepsilon_1, \ldots, f_r + \varepsilon_r$ as long as we take a sufficiently small perturbation. Huneke and Trivedi [27] extended this result for filter regular sequences, a generalization of the notion of regular sequence.

For numerical invariants, perhaps the most natural direction is to study the behavior of Hilbert function. Srinivas and Trivedi [62] showed that the Hilbert function of a sufficiently fine perturbation is at most the original Hilbert function. Furthermore they proved that the Hilbert functions of $R/(f_1,\dots,f_r)$ and $R/(f_1+\varepsilon_1,\dots,f_r+\varepsilon_r)$ coincide under small perturbations provided two conditions: (a) $\underline{f}=f_1,\dots,f_r$ a filter regular sequence; (b) $R/(f_1,\dots,f_r)$ is generalized Cohen-Macaulay. Recalling that $(R,\mathfrak{m})$ is generalized Cohen-Macaulay if all lower local cohomology $H^i_{\mathfrak{m}}(R), i<\dim R$, have finite length. Moreover, a generalized Cohen-Macaulay ring is Cohen-Macaulay on the punctured spectrum. Srinivas and Trivedi gave examples to show that the condition (a) is essential even if $\underline{f}=f_1,\dots,f_r$ is a part of system of parameters. However they asked whether the condition (b) is superfluous. Recently, Ma, Smirnov and Quy [35] answered affirmatively this question and proved the following.

Theorem 1.0.5. (see [35, Theorem 14].) *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension d, and $\underline{f}=f_1,\dots,f_r$ a filter regular sequence. Then there exists $N>0$ such that for every $\underline{\varepsilon}=\varepsilon_1,\dots,\varepsilon_r\in\mathfrak{m}^N$, the Hilbert functions of $R/(f_1,\dots,f_r)$ and $R/(f_1+\varepsilon_1,\dots,f_r+\varepsilon_r)$ are equal, i.e.*

$$\lambda(R/(\underline{f},\mathfrak{m}^n)=\lambda(R/(\underline{f_{\varepsilon}},\mathfrak{m}^n)$$

for all $n\geq 1$.[1]

We also asked the question.

Question 1.0.6. *Can one obtain explicit bounds on N?*

A certainly positive answer for the case $r=1$ was given in [35, Theorem 3.3]. If R is a Cohen-Macaulay local ring of dimension d, Srinivas and Trivedi [63, Proposition 1.1] provided a formula for N in terms of the multiplicity for any $r\geq 1$. Namely, we can choose

$$N=(d-r)!\,e(\mathfrak{m}R/(f_1,\dots,f_r))+2.$$

Inspired by the above formula, one can hope to give a bound for N in any local ring by using the extended degree instead of the multiplicity. See the next section for more details about the notion of extended degree. The aim on the present part is to give an evident for this belief. We will extend the above result of Srinivas and Trivedi for the class of generalized Cohen-Macaulay rings by using the multiplicity and the length of local cohomology $H^i_{\mathfrak{m}}(R)$.

Let $(R,\mathfrak{m})$ be a local ring and M a generalized Cohen-Macaulay module of dimension d. The Buchsbaum invariant of M is defined as follows

$$I(M)=\sum_{i=0}^{d-1}\binom{d-1}{i}\lambda(H^i_{\mathfrak{m}}(M)).$$

We now present the first main result of this part.

Theorem 1.0.7. (see Theorem 4.5.2.) *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay local ring of dimension d, and $\underline{f}=f_1,\dots,f_r$ a part of system of parameters of R. Let $s=d-r$, and*

$$N=s!\left(e(\mathfrak{m}R/(\underline{f}))+I(R/(\underline{f}))\right)+(s+1)I(R)+1.$$

Then for every $\underline{\epsilon}=\epsilon_1,\dots,\epsilon_r\in\mathfrak{m}^N$ we have the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_{\varepsilon}})$ are equal.

[1] Actually, we proved the result for any ideal J such that $(f_1,\dots,f_r)+J$ is $\mathfrak{m}$-primary. Although the main result of this paper can be extended for such ideals, we will keep our interest for the maximal ideal for simplicity.

The method of our proof of the above result is inspired by the Srinivas and Trivedi one in the Cohen-Macaulay case. Let us mention the most important step in our proof. If R is Cohen-Macaulay and $J = (x_1, \ldots, x_s)$ a minimal reduction of $\mathfrak{m}$ with respect to $R/(\underline{f})$, then we can choose N such that $J + (\underline{f}) = J + (\underline{f_\varepsilon})$ for any $\underline{\varepsilon} = \varepsilon_1, \ldots, \varepsilon_r \in \mathfrak{m}^N$. The strategy of Srinivas and Trivedi was to transform the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_\varepsilon})$ (with respect to $\mathfrak{m}$) to the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_\varepsilon})$ with respect to the parameter ideal J, and using the following well-known fact for Cohen-Macaulay rings

$$\lambda(R/(\underline{f}, J^{n+1})) = \binom{n+s}{s}\lambda(R/(\underline{f}, J)) = \binom{n+s}{s}\lambda(R/(\underline{f_\varepsilon}, J)) = \lambda(R/(\underline{f_\varepsilon}, J^{n+1})).$$

For generalized Cohen-Macaulay rings, we also have an explicit formula for the Hilbert function with respect to special parameter ideals, say standard parameter ideals, in terms of the length of lower local cohomology modules (see Theorem 4.1.6). Therefore we need to control $\lambda(H^i_{\mathfrak{m}}(R/(\underline{f}))$ under sufficiently small perturbations. This is the second main result of this part.

Theorem 1.0.8. (see Theorem 4.4.2.) *Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d and $\underline{f} = f_1, \ldots, f_r$ a part of system of parameters. Let $N = e(\mathfrak{m}R/(\underline{f}))+I(R)+1$, then for every $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$ we have*

$$\lambda(H^i_{\mathfrak{m}}(R/(\underline{f}))) = \lambda(H^i_{\mathfrak{m}}(R/(\underline{f_\varepsilon})))$$

for every $i < d - r$.

The results presented in this thesis are contained in the papers [64], [65] and [43]. All the computations in this thesis have been performed by using CoCoA [8] and Macaulay2 [22].

2. Fröberg's Conjecture

Fröberg's Conjecture is a longstanding conjecture on the Hilbert function of generic algebras which was introduced in [18]. Fröberg's Conjecture gives a formula for the Hilbert series of generic algebras. This problem is of central interest in commutative algebra in the last decades and a great deal was done (see for instance Anick [3], Fröberg [18], Fröberg-Hollman [19], Fröberg-Löfwall [20], Fröberg-Lundqvist [21], Moreno-Socías [33], Pardue [40], Stanley [60], Valla [67]). A large number of validations through computational methods push to a positive answer but the problem is still open. The main point of out approach is to pass through an equivalent conjecture stated by Pardue on the initial ideal of generic forms.

2.1. Hilbert Functions of graded algebras.

In this section we shall present the notion and some basic facts of the Hilbert Functions of graded algebras that we will use later. First we recall the notion of graded algebras.

Definition 2.1.1. A ring R is called graded (or more precisely, $\mathbb{Z}$-graded) if there exists a family of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R such that

(i) $R = \bigoplus_{n\in\mathbb{Z}} R_n$ (as abelian groups), and

(ii) $R_n.R_m \subseteq R_{n+m}$ for all n, m.

A non-zero element $x \in R_n$ is called a homogeneous element of R of degree n, denoted by $\deg(x) = n$. If $R_n = 0$ for all $n < 0$ and R is generated by elements of degree 1 over R_0 ($R = R_0[R_1]$), then we say R is homogeneous or standard graded.

Notice that if $R = \bigoplus_n R_n$ is a graded ring then R_0 is a subring of R.

Example 2.1.2. Let A be a ring and $x_1, \dots, x_k$ indeterminates over A. For $m = (m_1, \cdots, m_k) \in \mathbb{N}^k$, let $X^m = x_1^{m_1} \cdots x_k^{m_k}$. Then the polynomial ring $R = A[x_1, \dots, x_k]$ is a graded ring, where

$$R_n = \{\sum_{m\in\mathbb{N}^k} r_m X^m \mid r_m \in A \text{ and } m_1 + \cdots + m_k = n\}.$$

This is called the standard grading on R and it makes R into a standard graded ring. Notice that there are other useful gradings which can be put on R.

We have the definition of graded modules as follows:

Definition 2.1.3. Let R be a graded ring and M an R-module. M is called a graded R-module if there exists a family of subgroups $\{M_n\}_{n\in\mathbb{Z}}$ of M such that

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$ (as abelian groups), and

(ii) $R_n.M_m \subseteq M_{n+m}$ for all n, m.

Each M_n is called the n-th graded homogeneous component of M. If $u \in M \setminus \{0\}$ and $u = u_{i_1} + \ldots + u_{i_k}$ where $u_{i_j} \in R_{i_j} \setminus \{0\}$, then $u_{i_1}, \dots, u_{i_k}$ are called the homogeneous components of u.

Notice that if $M = \bigoplus_n M_n$ is a graded R-module then M_n is an R_0-module for all n.

Definition 2.1.4. Given any graded R-module M, we can form a new graded R-module by twisting the grading on M as follows: if n is any integer, define $M(n)$ (say M twisted by n) to be equal to M as an R-module, but with it's grading defined by $M(n)_k = M_{n+k}$. (For example, if $M = R(-3)$ then $1 \in M_3$.)

The class of graded (or homogeneous) submodules plays a very important role in the study of graded modules. It is defined as follows:

Definition 2.1.5. Let M be a graded R-module and N a submodule of M. For each $n \in \mathbb{Z}$, let $N_n = N \cap M_n$. If the family of subgroups $\{N_n\}$ makes N in to a graded R-module, we say that N is a graded submodule of M.

Remark 2.1.6. For any submodule N of M, we have $R_n.N_m \subseteq N_{n+m}$ for all n, m. Hence, N is homogeneous if and only if $N = \oplus_n N_n$. Moreover, by the definition of homogeneous submodules, one can check that the following statements are equivalent:

(1) N is a graded R-module.

(2) $N = \sum_n N \cap M_n$.

(3) For every $u \in N \setminus \{0\}$, all the homogeneous components of u are in N.

(4) N has a set of generators S such that for every $u \in S$, u is a homogenous element.

In particular, an ideal I of a graded ring R is homogeneous (or graded) if and only I has a homogeneous set of generators.

Proposition 2.1.7. *Let R be a graded ring, M a graded R-module and N a homogeneous submodule of M. Then M/N is a graded R-module, where*

$$\begin{aligned}(M/N)_n &= (M_n + N)/N \\ &= \{m + N \mid m \in M_n\}.\end{aligned}$$

Proof. Clearly, $\{(M/N)_n\}_n$ is a family of subgroups of M/N and

$$R_k.(M/N)_n = (R_k.M_n + N)/N \subseteq (M_{n+k} + N)/N = (M/N)_{n+k}.$$

Now, if $u \in M$ and $u = \sum_n u_n$ where $u_n \in M_n$ for each n, then $u + N = \sum_n (u_n + N)$. Thus $M/N = \sum_n (M/N)_n$. Finally, suppose $\sum_n (u_n + N) = 0 + N$ in M/N, where $u_n \in M_n$ for each n. Then $\sum_n u_n \in N$ and since N is a graded R-module, $u_n \in N$ for each n. Hence $u_n + N = 0 + N$ for all n and so $M/N = \sum_n (M/N)_n$ is an internal direct sum. □

In particular, if I is an homogeneous ideal in $R = K[x_1, \ldots, x_k]$ then R/I is a graded K-algebra with $(R/I)_n = R_n/I_n$ where $I_n = I \cap R_n$.

Example 2.1.8. Let $R = K[x, y, z]$ be the polynomial ring over field K and the grading on R is standard. Then $I = (x^2, x^3 + y^2z, y^4)$ is a homogenous ideal of R and R/I is a graded R-module (actually R-algebra), where

$$(R/I)_n = \{f + I \mid f \in R_n\}.$$

Definition 2.1.9. Let R be a graded ring and M, N graded R-modules. Let $f : M \mapsto N$ be an R-module homomorphism. Then f is said to be graded (or homogeneous) of degree d if $f(M_n) \subseteq N_{n+d}$ for all n.

The following is an elementary example of a graded homomorphism.

Example 2.1.10. Let M be a graded R-module and $r \in R_d$. Define $\mu_r : M \mapsto M$ by $\mu_r(m) = rm$ for all $m \in M$. Then μ_r is a graded homomorphism of degree d.

We now present the notion of the Hilbert function of graded modules. A graded R-module M is said to be bounded below if there exists $k \in \mathbb{Z}$ such that $M_n = 0$ for all $n \leq k$.

Definition 2.1.11. Let R be a graded ring and M a graded R-module. Suppose that $\lambda_{R_0}(M_n) < \infty$ for all n. We define the Hilbert function $H_M : \mathbb{Z} \mapsto \mathbb{Z}$ of M by

$$H_M(n) = \lambda_{R_0}(M_n)$$

for all $n \in \mathbb{Z}$. If in addition M is bounded below, we define Poincaré series (or Hilbert series) of M to be

$$P_M(z) = \sum_{n\in\mathbb{Z}} H_M(n) z^n$$

as an element of $\mathbb{Z}[|z|]$.

If M is a graded R-module such that $\lambda_{R_0}(M_n) < \infty$ for all n, we say that M has a Hilbert function or that the Hilbert function of M is defined. Similarly, if M is bounded below and has a Hilbert function, we say that M has a Poincaré series. The most important class of graded modules which have Hilbert functions are those which are finitely generated over a graded ring R, where R is Noetherian and R_0 is Artinian. On the other hand, if M is a finitely generated graded R-module which has a Hilbert function, then $R_0/Ann_{R_0}(M)$ is Artinian.

The following Proposition gives an example of a Hilbert function which, although very simple, provides an important prototype for all Hilbert functions.

Proposition 2.1.12. *Let $R = K[x_1, \ldots, x_d]$ be a polynomial ring over a field K and $\deg(x_1) = 1$ for $i = 1, \ldots, d$. Then*

$$H_R(n) = \binom{n+d-1}{d-1}$$

for all $n \geq 0$.

Proof. We use induction on $n + d$. The result is obvious if $n = 0$ or $d = 1$, so suppose $n > 0$ and $d > 1$. Let $S = K[x_1, \ldots, x_{d-1}]$ and consider the exact sequence

$$0 \to R_{n-1} \xrightarrow{x_d} R_n \to S_n \to 0.$$

Then

$$\begin{aligned} H_R(n) = \dim_K R_n &= \dim_K R_{n-1} + \dim_K S_n \\ &= \binom{n+d-2}{d-1} + \binom{n+d-2}{d-2} \\ &= \binom{n+d-1}{d-1}. \end{aligned}$$

□

Theorem 2.1.13. *Let R be a Noetherian graded ring and M a finitely generated graded R-module which has a Poincaré series. Then $P_M(z)$ is a rational function in z. In particular, if $R = R_0[x_1, \ldots, x_k]$ where R_0 is Artinian and $\deg(x_i) = s_i \neq 0$ then*

$$P_M(z) = \frac{h(z)}{\prod_{i=1}^{k}(1 - z^{s_i})}$$

where $h(z) \in \mathbb{Z}[z^{-1}, z]$.

Proof. If $k = 0$ then $R = R_0$. Since M is finitely generated, $M_n = 0$ for all but finitely many n. Thus, $P_M(z) \in \mathbb{Z}[z^{-1}, z]$. Suppose now that $k > 0$. Then consider the exact sequence

$$0 \to (0 :_M x_k)(-s_k) \to M(-s_k) \xrightarrow{x_k} M \to M/x_kM \to 0.$$

For each n, we have that

$$\lambda(M_n)z^n - \lambda(M_{n-s_k})z^n = \lambda((M/x_kM)_n)z^n - \lambda((0 :_M x_k)_{n-s_k})z^n.$$

Summing these equations over all $n \in \mathbb{Z}$, we obtain

$$P_M(z) - z^{s_k}P_M(z) = P_{M/x_kM}(z) - z^{s_k}P_{(0:_M x_k)}(z).$$

Since $x_kM/x_kM = 0$ and $x_k(0 :_M x_k) = 0$, M/x_kM and $(0 :_M x_k)$ are modules over $R_0[x_1, \ldots, x_{k-1}]$. Since M is bounded below, so are M/x_kM and $(0 :_M x_k)$. By induction, $P_{M/x_kM}(z)$ and $P_{(0:_M x_k)}(z)$ are of the required form, and so there exists $h_1(z), h_2(z) \in \mathbb{Z}[z^{-1}, z]$ such that

$$(1 - z^{s_k})P_M(z) = \frac{h_1(z)}{\prod_{i=1}^{k-1}(1 - z^{s_i})} - \frac{z^{s_k}h_2(z)}{\prod_{i=1}^{k-1}(1 - z^{s_i})}.$$

Dividing by $(1 - z^{s_k})$, we obtain the desired result. □

An important special case is given by the following corollary:

Corollary 2.1.14. *Let $R = R_0[x_1, \ldots, x_k]$ be a Noetherian graded ring where R_0 is Artinian and $\deg(x_i) = 1$ for all i. Let M be a non-zero finitely generated graded R-module. Then there exists a unique integer $s = s(M)$ with $0 \le s \le k$ such that*

$$P_M(z) = \frac{h(z)}{(1 - z)^s}$$

for some $h(z) \in \mathbb{Z}[z^{-1}, z]$ with $h(1) \neq 0$.

Proof. By Theorem 2.1.13, we have that $P_M(z) = \frac{f(z)}{(1-z)^k}$ for some $f(z) \in \mathbb{Z}[z^{-1}, z]$. We can write $f(z) = (1 - z)^m h(z)$ where $m \ge 0$ and $h(1) \neq 0$. Let $s = k - m$ then we are done if $s \ge 0$. But if $s < 0$ then $P_M(z) \in \mathbb{Z}[z^{-1}, z]$ and $P_M(1) = 0$. Hence $\sum_n \lambda(M_n) = 0$ and so $M = 0$, contrary to our assumption. The uniqueness of s and $h(z)$ is clear. □

Proposition 2.1.15. *Let M be a graded R-module having a Poincaré series of the form*

$$P_M(z) = \frac{f(z)}{(1 - z)^s}$$

for some $s \ge 0$ and $f(z) \in \mathbb{Z}[z^{-1}, z]$ with $f(1) \neq 0$. Then there exists a unique polynomial $p_M(x) \in \mathbb{Q}[x]$ of degree $s - 1$ such that $H_M(n) = p_M(n)$ for all sufficiently large integers n.

Proof. Let $f(z) = a_l z^l + a_{l+1}z^{l+1} + \ldots + a_m z^m$. We have

$$\begin{aligned} P_M(z) &= \frac{f(z)}{(1 - z)^s} \\ &= f(z).\sum_{n=0}^{\infty} \binom{n + s - 1}{s - 1} z^n. \end{aligned}$$

Comparing coefficients of z^n, we see that for $n \ge m$

$$H_M(n) = \sum_{i=l}^{m} a_i \binom{n + s - i - 1}{s - 1}.$$

Let $p_M(x) = \sum_i a_i \binom{x+s-i-1}{s-1}$. Then $p_M(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree at most $s-1$ and $p_M(n) = H_M(n)$ for all n sufficiently large. Note that the coefficient of x^{s-1} is $(a_l + \ldots + a_m)/(s-1)! = f(1)/(s-1)! \neq 0$. Thus $\deg p_M(x) = s-1$. $\square$

Definition 2.1.16. Let R be a graded ring and M a graded R-module which has a Hilbert function $H_M(n)$. A polynomial $p_M(x) \in \mathbb{Q}[x]$ is called the Hilbert polynomial of M if $p_M(n) = H_M(n)$ for all sufficiently large integers n.

The following corollary follows immediately from Corollary 2.1.14 and Proposition 2.1.15.

Corollary 2.1.17. *Let $R = R_0[x_1, \ldots, x_k]$ be a Noetherian graded ring where R_0 is Artinian and $\deg(x_i) = 1$ for all i. Let M be a non-zero finitely generated graded R-module. Then M has a Hilbert polynomial $p_M(x)$ and $\deg p_M(x) = s(M) - 1 \leq k-1$.*

Notice that if $\dim(M) = d$, then $\deg p_M(x) = d-1$ (see for instance [7, Theorem 4.1.3]). Since $p_M(n) \in \mathbb{Z}$ for sufficiently large n, it follows that $p_M(\mathbb{Z}) \subseteq \mathbb{Z}$. By [7, Lemma 4.1.4], there exist unique integers $e_i = e_i(M)$ for $i = 0, \ldots, d-1$ such that

$$p_M(x) = e_0 \binom{x+d-1}{d-1} - e_1 \binom{x+d-2}{d-2} + \ldots + (-1)^{d-1} e_{d-1}.$$

The integers $e_0, \ldots, e_{d-1}$ are called the Hilbert coefficients of M. The first coefficient, e_0, is called the multiplicity of M and is denoted $e(M)$. In the case $\dim M = 0$, $e(M)$ is defined to be $\lambda(M)$.

2.2. **Fröberg's Conjecture.** Let $R = K[x_1, \cdots, x_n]$ be the polynomial ring in n variables over an infinite field K such that $\deg x_i = 1$ for all i. The Hilbert function of the standard graded algebra $A = R/I$ where I is an homogeneous ideal of R is by definition (see Definition 2.1.11)

$$H_A(t) := \dim_K R_t/I_t \text{ for every } t \geq 0,$$

where $(\ )_t$ denotes the homogeneous part of degree t.

A homogeneous ideal I in R is said to be of type $(n; d_1, \cdots, d_r)$ if it is minimally generated by r forms of degree d_i for $i = 1, \cdots, r$. We are interested in the behavior of the Hilbert function of generic ideals of type $(n; d_1, \cdots, d_r)$. Here we adopt the definition of generic ideals given by Fröberg because it is more suitable for our approach. Assume that K is an extension of a base field F then *generic ideals* are defined by Fröberg in [18] as the following.

Definition 2.2.1. A form of degree d in $K[x_1, \cdots, x_n]$ is called *generic* over K if it is a linear combination of all monomials of degree d and all coefficients are algebraically independent over K. A homogeneous ideal $(f_1, \cdots, f_r)$ is called generic if all f_i are generic forms and all coefficients appeared in $f_1, \cdots, f_r$ are algebraically independent over K.

The Hilbert function of a generic ideal of type $(n; d_1, \cdots, d_n)$ is the Hilbert function of a regular sequence, hence the generating Hilbert series $P_A(z) := \sum_{t \geq 0} H_A(t) z^t$ is well known and $P_A(z) = \dfrac{\prod_{i=1}^{n}(1 - z^{d_i})}{(1-z)^n}$.

We present here the intrinsic motivation which leads to Fröberg's conjecture. In general if A is a graded standard K-algebra and $f \in R_d$ is a generic form, it is natural to guess that for every $t \geq 0$, the multiplication map

$$A_t \xrightarrow{f} A_{t+d}$$

is of maximal rank (either injective or surjective), hence

$$H_{A/fA}(t) = \max\{0, H_A(t) - H_A(t-d)\}.$$

In terms of the corresponding Hilbert series this can be rewritten as

$$P_{A/fA}(z) = \left\lceil (1-z^j)P_A(z) \right\rceil,$$

where for a power series $\sum a_i z^i$ one has $\lceil \sum a_i z^i \rceil = \sum b_i z^i$, with $b_i = a_i$ if $a_j > 0$ for all $j \leq i$, and $b_i = 0$ otherwise. Following the terminology of Pardue in [40], if this is the case, we say that f is *semi-regular*. A sequence of homogeneous polynomials $f_1, \cdots, f_r$ is a *semi-regular sequence* on A if each f_i is semi-regular on $A/(f_1, \cdots, f_{i-1})$. Clearly regular sequences are semi-regular sequences. Actually in 1985 Fröberg conjectured that generic forms are semi-regular sequences.

Conjecture 2.2.2. (Fröberg's Conjecture.) *Let $I = (f_1, \cdots, f_r)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_r)$ in $R = K[x_1, \cdots, x_n]$. Then the Hilbert series of $A = R/I$ is given by*

$$P_A(z) = \left\lceil \frac{\prod_{i=1}^{r}(1-z^{d_i})}{(1-z)^n} \right\rceil.$$

2.3. **Initial ideals and Gröbner bases.** The notion of Gröbner bases plays a very important role in the graded algebras. In this section we recall the definition of Gröbner bases and the Buchberger's Algorithm for computing Gröbner bases. For general facts and properties on Gröbner bases see for instance [15, Section 15].

Let $R = K[x_1, \ldots, x_d]$ be a polynomial ring over a field K and $\deg(x_1) = 1$ for $i = 1, \ldots, d$. A *monomial* m in R is an element of the form $m = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ where $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. A *term* in R is a monomial multiplied by a scalar in K. It is well known that the monomials form a vector space basis for R and every polynomial $f \in R$ is uniquely expressible as a finite sum of nonzero terms involving distinct monomials, which we call the terms of f; the monomials in these terms will be called the monomials of f.

Definition 2.3.1. A *monomial order* on R is a total order $>$ on the monomials of R such that if m_1, m_2 are monomials of R and $n \neq 1$ is a monomial of R, then

$$m_1 > m_2 \Rightarrow nm_1 > nm_2 > m_2.$$

This notation can be extended to terms: If um and vn are terms with $0 \neq u, v \in K$, and m, n are monomials with $m > n$ then we say $um > vn$.

Example 2.3.2. Let $m = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $n = x_1^{\beta_1} \ldots x_n^{\beta_n}$ are monomials. We always assume that $x_1 > x_2 > \ldots > x_n$ in the following orders.

(i) Lexicographic order. $m >_{lex} n$ iff $\alpha_i > \beta_i$ for the first index i with $\alpha_i \neq \beta_i$.

(ii) Degree lexicographic order. $m >_{dlex} n$ iff $\deg m > \deg n$ or $\deg m = \deg n$ and $\alpha_i > \beta_i$ for the first index i with $\alpha_i \neq \beta_i$.

(iii) Degree reverse lexicographic order. $m >_{rlex} n$ iff $\deg m > \deg n$ or $\deg m = \deg n$ and $\alpha_i < \beta_i$ for the last index i with $\alpha_i \neq \beta_i$.

We can now define the notion of the initial terms and the initial ideals as well. Let $>$ be a monomial order on R, then for any $f \in R$ the *initial term* of f, denoted by $\mathrm{in}_>(f)$, is the greatest term of f with respect to the given order. The monomial of the term $\mathrm{in}_>(f)$ is called the *leading monomial* of f. If I is an ideal of R then the *initial ideal* of I, denoted by $\mathrm{in}_>(I)$, is the ideal of R defined by

$$\mathrm{in}_>(I) := (\{\mathrm{in}_>(f) \mid f \in I\}).$$

It is clearly that the $\mathrm{in}_>(I)$ is a monomial ideal of R. When there is no confusion we will simply write $\mathrm{in}(f)$ (respectively, $\mathrm{in}(I)$) in place of $\mathrm{in}_>(f)$ (respectively, $\mathrm{in}_>(I)$). The following theorem due to Macaulay gives a very useful application of the initial ideal.

Theorem 2.3.3. [15, Theorem 15.3] *Let I be an arbitrary ideal of R. For any monomial order $>$ on R, the set B of all monomials not in $\mathrm{in}_>(I)$ forms a basis for R/I.*

We now define the main notion of this section that is the Gröbner basis.

Definition 2.3.4. A *Gröbner basis* with respect to an order $>$ on R is a set of elements $g_1, \ldots, g_t \in R$ such that if I is the ideal of R generated by $g_1, \ldots, g_t$, then $\mathrm{in}_>(g_1), \ldots, \mathrm{in}_>(g_t)$ generate $\mathrm{in}_>(I)$. We then say that $g_1, \ldots, g_t$ is a Gröbner basis for I. A Gröbner basis $g_1, \ldots, g_t$, which consists of monic polynomials, is called reduced if for any g_i, g_j with $i \neq j$, the leading monomial of g_i does not divide any monomial of g_j.

The following proposition is the main tool for the Buchberger's Algorithm.

Proposition 2.3.5. *Let $>$ be an order on R. If $f, g_1, \ldots, g_t \in R$ then there is an expression*

$$f = \sum_{i=1}^{t} f_i g_i + f',$$

where $f_i, f' \in R$ such that none of the monomials of f' is in $(\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t))$ and

$$\mathrm{in}(f) \geq \mathrm{in}(f_i g_i)$$

for every $i = 1, \ldots, t$.

With this notation we have the following definition.

Definition 2.3.6. Any such f' is called a *remainder* (or *reduced form*) of f with respect to $g_1, \ldots, g_t$, and an expression $f = \sum_{i=1}^{t} f_i g_i + f'$ satisfying the condition of Proposition 2.3.5 is called a *standard expression* for f in terms of the g_i.

We can find the remainder of f with respect to $g_1, \cdots, g_t$ by Division Algorithm, which was introduced in [15, Section 15], as follows:

If none of the monomials of f is in $(\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t))$ then f is the remainder of f with respect to $g_1, \cdots, g_t$. Otherwise let m_0 is the maximal term of f that is divisible by some $\mathrm{in}(g_{u_0})$, where $u_0 \in \{1, 2, \ldots, t\}$. Set

$$f_1 := f - \frac{m_0}{\mathrm{in}(g_{u_0})} g_{u_0}.$$

If $f_1 = 0$ or none of the monomials of f_1 is in $(\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t))$ then f_1 is the remainder of f with respect to $g_1, \cdots, g_t$. Otherwise let m_1 is the maximal term of f_1 that is divisible by some $\mathrm{in}(g_{u_1})$, where $u_1 \in \{1, 2, \ldots, t\}$, and set

$$f_2 := f_1 - \frac{m_1}{\mathrm{in}(g_{u_1})} g_{u_1}.$$

Continuing this process untill we find a polynomial

$$f_k := f_{k-1} - \frac{m_{k-1}}{\mathrm{in}(g_{u_{k-1}})} g_{u_{k-1}}$$

such that $f_k = 0$ or none of the monomials of f_k is in $(\mathrm{in}(g_1), \ldots, \mathrm{in}(g_t))$. Then f_k is the remainder of f with respect to $g_1, \cdots, g_t$.

We now present the Buchberger's Algorithm. Let $I = (g_1, \ldots, g_t)$ be an ideal of R.

• Step 1. For each distinct pair of i, j where $1 \leq i, j \leq t$ we set

$$\sigma_{ij} = \frac{\mathrm{in}(g_j)g_i - \mathrm{in}(g_i)g_j}{GCD(\mathrm{in}(g_i), \mathrm{in}(g_j))} \in R,$$

where GCD is the greatest common divisor.

• Step 2. Compute a remainder of σ_{ij} with respect to $g_1, \ldots, g_t$, say h_{ij}.

• Step 3. If $h_{ij} = 0$ for all pair of i, j then $g_1, \ldots, g_t$ forms a Gröbner basis for I (see [15, Theorem 15.8]). If some $h_{ij} \neq 0$, then replace $g_1, \ldots, g_t$ with $g_1, \ldots, g_t, h_{ij}$, and repeat the process.

As the ideal generated by the initial forms of $g_1, \ldots, g_t, h_{ij}$ is strictly larger than that generated by the initial forms of $g_1, \ldots, g_t$, this process must terminate after finitely many steps.

Example 2.3.7. Let $R = \mathbb{Q}[x_1, x_2]$ and $I = (x_1^2 + x_2, x_1x_2 + x_2^2)$ an ideal of R. We now apply the Buchberger's Algorithm to find a Gröbner basis for I with respect to the Degree reverse lexicographic order.

Let $g_1 = x_1^2 + x_2$ and $g_2 = x_1x_2 + x_2^2$ we have $\mathrm{in}(g_1) = x_1^2$ and $\mathrm{in}(g_2) = x_1x_2$. The GCD of $\mathrm{in}(g_1)$ and $\mathrm{in}(g_2)$ is x_1 and $\sigma_{12} = -x_1x_2^2 + x_2^2$. Since

$$\sigma_{12} = -x_2g_2 + x_2^3 + x_2^2,$$

by Proposition 2.3.5 we have the remainder of σ_{12} with respect to g_1, g_2 is $h_{12} = x_2^3 + x_2^2$. So that we replace g_1, g_2 with g_1, g_2, g_3 where $g_3 = h_{12} = x_2^3 + x_2^2$. Now it is clear that the remainder of σ_{12} with respect to g_1, g_2, g_3 is 0. Next we have $\mathrm{in}(g_3) = x_2^3$ and

$$\sigma_{13} = -x_1^2x_2^2 + x_2^4 = -x_2^2g_1 + x_2g_3.$$

Hence the remainder of σ_{13} with respect to g_1, g_2, g_3 is 0. Similarly, we have

$$\sigma_{23} = x_2^4 - x_1x_2^2 = -x_2g_2 + x_2g_3,$$

so that the remainder of σ_{23} with respect to g_1, g_2, g_3 is 0 as well. Thus, g_1, g_2, g_3 forms a Gröbner basis for I and we have $\mathrm{in}(I) = (x_1^2, x_1x_2, x_2^3)$.

By using a device related to the Gröbner basis, Valla [67] gave a proof of Fröberg's Conjecture in the case $R = K[x_1, x_2]$. The crucial information in the Valla's approach comes from an old result by A. Galligo (see [5]). Here $Gl(n, k)$ is the general linear group acting on R and its Borel subgroup is the subgroup

$$\mathcal{B} := \{g \in Gl(n, k) \mid g_{ij} = 0 \ \forall j < i\}.$$

Theorem 2.3.8. (***Galligo***) *Let I be an homogeneous ideal of R; there exists a Zariski open set $\mathcal{U} \subseteq Gl(n, k)$, such that the monomial ideal $\mathrm{in}(gI)$ is invariant under the action of $\mathcal{B}$.*

Now it is easy to see that if J is a monomial ideal in R then J is Borel-fixed if and only if the condition $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in J$ implies $x_1^{\alpha_1} \ldots x_j^{\alpha_j + q} \ldots x_i^{\alpha_i - q} \ldots x_n^{\alpha_n} \in J$ for every j, i and q such that $1 \leq j < i \leq n$ and $0 \leq q \leq \alpha_i$.

If we restrict ourselves to the case when $n = 2$, it is clear that the ideal J is Borel-fixed if and only if J as a K-vector space is generated in each degree by an initial segment in the given order.

We come now to the proof of Fröberg's Conjecture in $K[x_1, x_2]$.

Theorem 2.3.9. (see [67, Theorem 4.3]) *Let $f_1, \ldots, f_r$ be generic forms of degree $d_1, \ldots, d_r$ in $R = K[x_1, x_2]$. If $I = (f_1, \ldots, f_r)$, then*

$$P_{R/I}(z) = \left\lceil \frac{\prod_{i=1}^{r}(1 - z^{d_i})}{(1-z)^2} \right\rceil .$$

Proof. Since theorem is true if $r \leq 2$, we may argue by induction on r and assume that the graded algebra $B := R/(f_1, \ldots, f_{r-1})$ is given with the expected Hilbert series. We must show that we can find a form f of degree $d := d_r$ in R such that

$$B_t \xrightarrow{f} B_{t+d}$$

is of maximal rank. We fix in the set of monomials of R the degree reverse lexicographic order. Let $J = (f_1, \ldots, f_{r-1})$; since the generators are generic we have that $\mathrm{in}(J)$ is Borel-fixed. Hence, for every degree t, if $H_{R/J}(t) = k$, we have that $x_1^t, x_1^{t-1}x_2, \ldots, x_1^k x_2^{t-k}$ generates $\mathrm{in}(J)_t$ and $x_1^{k-1}x_2^{t-k+1}, \ldots, x_2^t$ is a K-base of B_t.

We let $d := d_r$ and $f = f_r = x_2^d$. We have two possibilities. Either $k := \dim B_t < \dim B_{t+d}$ or $k := \dim B_t \geq \dim B_{t+d}$. In the first case, by Macaulay's theorem, we must have $k = t+1$, so that a K-base of B_t is

$$\{x_1^t, x_1^{t-1}x_2, \ldots, x_1 x_2^{t-1}, x_2^t\}.$$

By multiplying these monomials with x_2^d, we get the smallest $t+1$ monomials of degree $t+d$. Since $\dim B_{t+d} > t+1$, these monomials are linearly independent being part of a K-base of B_{t+d}.

In the second case, a K-base of B_t is

$$\{x_1^{k-1}x_2^{t-k+1}, \ldots, x_2^t\}$$

By multiplying these monomials with x_2^d, we get the smallest k monomials of degree $t+d$. Since $\dim B_{t+d} \leq k$, the conclusion follows. $\square$

2.4. **Moreno-Socías and Pardue's Conjectures.** Since the Hilbert series of R/I and of $R/\mathrm{in}_>(I)$ coincide for every monomial order $>$, a rich literature has been developed with the aim to characterize the initial ideal of generic ideals with respect to suitable term orders (see [1], [3], [11], [10], [12], [30], [32], [33], [40]). From now on, the initial ideal of I will be always with respect to the *degree reverse lexicographic order* and it will be denoted simply by $\mathrm{in}(I)$. For general facts and properties on the degree reverse lexicographic order see for instance [15, Proposition 15.2]. Moreno-Socías stated a conjecture describing the initial ideal of generic forms in the case $r = n$. It is natural to guess generic complete intersections share *special* initial ideals. We present here Moreno-Socías' Conjecture.

Conjecture 2.4.1. (see Moreno-Socías [32].) *Let $I = (f_1, \cdots, f_n)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $K[x_1, \cdots, x_n]$. Then $\mathrm{in}(I)$ is almost reverse lexicographic, i.e, if x^μ is a minimal generator of $\mathrm{in}(I)$ then every monomial of the same degree and greater than x^μ must be in $\mathrm{in}(I)$ as well.*

Moreno-Socías' Conjecture was proven in the case $n = 2$ by Aguire et al. [1] and Moreno-Socías [33], $n = 3$ by Cimpoeas [12], $n = 4$ by Harima and Wachi [30], and sequences $d_1, \cdots, d_n$ satisfying $d_i > \sum_{j=1}^{i-1} d_j - i + 1$ for every $i \geq 4$ by Cho and Park assuming $\mathrm{char} K = 0$ [10]. Without restriction on the characteristic of K, by using an incremental method from [23], Capaverde and Gao improved the result of Cho and Park. They gave a complete description of the initial ideal of I in the case $d_1, \cdots, d_n$ satisfies $d_i \geq \sum_{j=1}^{i-1} d_j - i - 1$ for every i. In particular they proved Moreno-Socías' Conjecture under the above conditions, see [11, Theorem 3.19]. Moreno-Socías' Conjecture (proved in its full generality) implies Fröberg's Conjecture. Pardue stated a conjecture which is equivalent to Fröberg's Conjecture ([40, Theorem 2]) by a slight modification of the requirement on the initial ideal.

For every monomial $x^\alpha \in K[x_1, \cdots, x_n]$, denote by $\max(x^\alpha)$ the largest index i such that x_i divides x^α.

Conjecture 2.4.2. (Pardue's Conjecture.) *Let $I = (f_1, \cdots, f_n)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $K[x_1, \cdots, x_n]$. If x^μ is a minimal generator of $\mathrm{in}(I)$ and $\max(x^\mu) = m$ then every monomial of the same degree in the variables $x_1, \cdots, x_{m-1}$ must be in $\mathrm{in}(I)$ as well.*

2.5. **Incremental method for computing Gröbner bases.** In this section we recall an effective method for computing Gröbner bases that was introduced by Gao, Guan and Volny in [23], say *incremental method.*

Let I be an ideal of R and g an element in R. Suppose that we know a Gröbner basis G for I with respect to some monomial order $>$. Then the incremental method gives us a Gröbner basis for ideal $I + (g)$.

Let $B = \{x^{\alpha_1} = 1, x^{\alpha_2}, \ldots, x^{\alpha_N}\}$ be the set of all the monomials that are not in $\mathrm{in}(I)$ and assume that the monomials in B are ordered in increasing order, that is, $x^{\alpha_j} > x^{\alpha_i}$ whenever $i < j$. Note that when I is not zero-dimensional, we have N is ∞ and it is possible that there are infinitely many monomials between two monomials in B (especially for lexicographic order). For each $1 \leq i \leq N$, by Proposition 2.3.5, we can choose a remainder h_i of $x^{\alpha_i} g$ with respect to Gröbner basis G. We write this as follows:

$$\begin{pmatrix} x^{\alpha_1} \\ x^{\alpha_2} \\ \vdots \\ x^{\alpha_N} \end{pmatrix} .g \equiv \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{pmatrix} \pmod G \tag{1}$$

Note that each h_i is a K-linear combination of monomials in B. Suppose x^β is the leading monomial of h_1. We apply the row operations to both sides of Equation (1) in order to eliminate the monomial x^β in all h_j, $2 \leq j \leq N$. In fact, we only need to eliminate x^β if it is the leading monomial. Then continue with the leading monomial of the polynomial in the second row of the resulting and so on. Since a monomial order is a well ordering, there is no infinite decreasing sequence of monomials, hence each h_i needs only be reduced by finitely many rows above it (even if there are infinitely many rows about the row of h_i). Thus, after the row operations as above, the right hand side of Equation (1) can be transformed into a quasi-triangular form, and the Equation (1) has the following form.

$$
\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} .g \equiv \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} \pmod{G} \tag{2}
$$

where $u_i, v_i \in R$ are K-linear combinations of monomials in B, and for $1 \leq i < j \leq N$ with $v_i, v_j \neq 0$, we have the leading monomials of v_i and v_j are different, that is, the nonzero rows in the right-hand side of Equation (2) have distinct leading monomials.

Finally, from the following theorem we get the Gröbner basis for ideal $I + (g)$.

Theorem 2.5.1. [23, Section 2]. *Let $\widetilde{G} = G \cup \{v_i \mid 1 \leq i \leq N\}$. Then $\widetilde{G}$ is a Gröbner basis for (I, g).*

Example 2.5.2. Let $R = \mathbb{Q}[x_1, x_2]$ and $I = (x_1^2 + x_2, x_1x_2 + x_2^2)$ an ideal of R. In Example 2.3.7, we know that $G = \{g_1, g_2, g_3\}$, where $g_1 = x_1^2 + x_2, g_2 = x_1x_2 + x_2^2, g_3 = x_2^3 + x_2^2$, forms a Gröbner basis for I with respect to the Degree reverse lexicographic order and $\text{in}(I) = (x_1^2, x_1x_2, x_2^3)$.

Let $g = x_2^2 - 3x_1 + 2 \in R$. We now apply the Incremental method to find a Gröbner basis for $I + (g)$. First we have the set of all the monomials that are not in $\text{in}(I)$ is the following

$$B = \{1, x_2, x_1, x_2^2\}.$$

One can compute the remainders of the polynomials g, x_2g, x_1g, x_2^2g with respect to Gröbner basis G are $h_1 = g, h_2 = 2x_2^2 + x_2, h_3 = x_2^2 + 3x_2 + 2x_1, h_4 = 0$ respectively. We write this as follows:

$$
\begin{pmatrix} 1 \\ x_2 \\ x_1 \\ x_2^2 \end{pmatrix} .g \equiv \begin{pmatrix} x_2^2 - 3x_1 + 2 \\ 2x_2^2 + x_2 \\ x_2^2 + 3x_2 + 2x_1 \\ 0 \end{pmatrix} \pmod{G}
$$

Applying the row operations we get

$$
\begin{pmatrix} x_2^2 - 3x_1 + 2 \\ 2x_2^2 + x_2 \\ x_2^2 + 3x_2 + 2x_1 \\ 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} x_2^2 - 3x_1 + 2 \\ 6x_1 + 2x_2 - 4 \\ 5x_1 + 3x_2 - 2 \\ 0 \end{pmatrix} \dashrightarrow \begin{pmatrix} x_2^2 - 3x_1 + 2 \\ 6x_1 + 2x_2 - 4 \\ \frac{4}{3}x_2 + \frac{4}{3} \\ 0 \end{pmatrix}
$$

Thus, $\widetilde{G} = G \cup \{v_1, v_2, v_3\}$, where $v_1 = x_2^2 - 3x_1 + 2, v_2 = 6x_1 + 2x_2 - 4, v_3 = \frac{4}{3}x_2 + \frac{4}{3}$ forms a Gröbner basis for $I + (g)$.

2.6. **Gröbner basis of generic ideals.** In this section we give a Gröbner basis of generic ideals by using the Incremental method.

Let $R' = K[x_1, \cdots, x_n, z]$ be the polynomial ring in $n + 1$ variables and fix the order on the variables $x_1 > \cdots > x_n > z$. Let $(I, g) = (f_1, \cdots, f_n, g)$ be a generic homogeneous ideal of type $(n + 1; d_1, \cdots, d_n, d)$ in R'. Note that $I = (f_1, \cdots, f_n)$ is a generic homogeneous ideal of type $(n + 1; d_1, \cdots, d_n)$ in R'. Define $\pi : R' \longrightarrow R = K[x_1, \cdots, x_n]$ to be the ring homomorphism where z goes to zero, fixing the elements in K and the variables $x_1, \cdots, x_n$. Let $J = \pi(I)$ be the image of I. Then J is a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in R.

Proposition 2.6.1. $\text{in}(I)$ *and* $\text{in}(J)$ *have the same minimal generators.*

Proof. From a property of the degree reverse lexicographic order in [15, Proposition 15.12], we get $\pi(\mathrm{in}(I)) = \mathrm{in}(J)$. On the other hand, since z is regular in R'/I, by [15, Theorem 15.13] z is regular in $R'/\mathrm{in}(I)$. Furthermore, by [15, Theorem 15.14] the minimal generators of $\mathrm{in}(I)$ are not divisible by z. Thus, $\mathrm{in}(I)$ and $\mathrm{in}(J)$ have the same minimal generators. $\square$

Let $B = B(J)$, which is called the set of standard monomials with respect to J, be the set of monomials in R that are not in $\mathrm{in}(J)$. We set

$$\delta = \delta_n = d_1 + \cdots + d_n - n,$$

$$\sigma = \sigma_n = \min\left\{\delta_{n-1}, \left\lfloor \frac{\delta_n}{2} \right\rfloor\right\}.$$

It is known that $A = R/J$ is a complete intersection and the Hilbert series of A is a symmetric polynomial of degree δ, say $P_A(z) = \sum_{i=0}^{\delta} a_i z^i$, with

$$0 < a_0 < a_1 < \cdots < a_\sigma = \cdots = a_{\delta-\sigma} > \cdots > a_{\delta-1} > a_\delta > 0.$$

Notice that $a_i = |B_i|$ where B_i is the set of monomials of degree i in B (see for instance [33, Proposition 2.2]).

The set of standard monomials with respect to a generic ideal of type $(n; d_1, \cdots, d_r)$ depends only on $(n; d_1, \cdots, d_r)$. So, we denote it by $B(n; d_1, \cdots, d_r)$. We will describe more clearly the set of standard monomials in the case $r = n$ which is $B = B(n; d_1, \cdots, d_n)$. For each $1 \le i \le \sigma$, define

$$\widetilde{B}_i^0 = \{x^\mu \in B_i \mid x_n \text{ does not divide } x^\mu\}.$$

Proposition 2.6.2. *The structure of $B = B(n; d_1, \cdots, d_n)$ is as follows,*

(1) $B_i = \widetilde{B}_i^0 \cup x_n \widetilde{B}_{i-1}^0 \cup \cdots \cup x_n^{i-1} \widetilde{B}_1^0 \cup \{x_n^i\}$

$= \widetilde{B}_i^0 \cup x_n B_{i-1}$, *for all* $1 \le i \le \sigma$.

(2) $B_{\sigma+i} = x_n^i \widetilde{B}_\sigma^0 \cup x_n^{i+1} \widetilde{B}_{\sigma-1}^0 \cup \cdots \cup x_n^{\sigma+i-1} \widetilde{B}_1^0 \cup \{x_n^{\sigma+i}\}$

$= x_n^i B_\sigma$, *for all* $0 \le i \le \delta - 2\sigma$.

(3) $B_{\delta-i} = x_n^{\delta-2i} \widetilde{B}_i^0 \cup x_n^{\delta-2i+1} \widetilde{B}_{i-1}^0 \cup \cdots \cup x_n^{\delta-i-1} \widetilde{B}_1^0 \cup \{x_n^{\delta-i}\}$

$= x_n^{\delta-2i} B_i$, *for all* $0 \le i \le \sigma$.

Proof. (1) For $1 \le i \le \sigma$, we have $a_{i-1} < a_i$. Let S denote the subset of B_i consisting of the a_{i-1} smallest monomials in B_i with respect to the degree reverse lexicographic order . By [11, Lemma 3.5] we get $S = x_n B_{i-1}$. It is clear that $\widetilde{B}_i^0 \subseteq B_i \setminus S$. Conversely, for every monomial $x^\alpha \in B_i \setminus S$, assume that $x^\alpha \notin \widetilde{B}_i^0$. Then x_n divides x^α, so that $x^\alpha / x_n \in B_{i-1}$. This implies a contradiction since $x_\alpha = x_n(x^\alpha/x_n) \in S$. Thus, $\widetilde{B}_i^0 = B_i \setminus S$ and (1) holds.

(2) For $0 \le i \le \delta - 2\sigma$, we have $a_{\sigma+i} = a_\sigma$. By [11, Lemma 3.5] we get $B_{\sigma+i} = x_n^i B_\sigma$.

(3) Since $\sigma \leq \dfrac{\delta}{2}$, by [11, Lemma 3.4] we get $B_{\delta-i} = x_n^{\delta-2i} B_i$ for $0 \leq i \leq \sigma$. $\square$

Remark 2.6.3. (1) $B = B(n; d_1, \cdots, d_n)$ is determined by $\widetilde{B}_1^0, \cdots, \widetilde{B}_\sigma^0$.

(2) $|\widetilde{B}_i^0| = a_i - a_{i-1} = a_i' > 0$, for all $1 \leq i \leq \sigma$.

In the following example, we explicitly compute $B(4; 2, 3, 3, 4)$ for the reader according to Proposition 2.6.2.

Example 2.6.4. Let $B = B(4; 2, 3, 3, 4)$ be the set of standard monomials with respect to a generic ideal of type $(4; 2, 3, 3, 4)$. Then $\delta = 8; \sigma = 4$ and we have
$B_0 = \{1\}$,
$B_1 = \widetilde{B}_1^0 \cup \{x_4\}$ where $\widetilde{B}_1^0 = \{x_1, x_2, x_3\}$,
$B_2 = \widetilde{B}_2^0 \cup x_4 B_1$ where $\widetilde{B}_2^0 = \{x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$,
$B_3 = \widetilde{B}_3^0 \cup x_4 B_2$ where $\widetilde{B}_3^0 = \{x_1x_2x_3, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_3^3\}$,
$B_4 = \widetilde{B}_4^0 \cup x_4 B_3$ where $\widetilde{B}_4^0 = \{x_2x_3^3, x_3^4\}$,
$B_5 = x_4^2 B_3$,
$B_6 = x_4^4 B_2$,
$B_7 = x_4^6 B_1$,
$B_8 = x_4^8 B_0$.

In the end of this section, we will use the above example to construct $B(n+1, d_1, \cdots, d_{n+1})$ starting from $B(n, d_1, \cdots, d_n)$.

We now using the incremental method and adapted to our situation in [11] to construct a Gröbner basis of ideal $(I, g) = (f_1, \cdots, f_n, g)$ starting from $\mathrm{in}(I)$ and $B(n, d_1, \cdots, d_n)$. Define C_I to be the set of the coefficients of the polynomials $f_1, \cdots, f_n$ and $\bar{F} = F(C_I) \subset K$, where K is an extension of a base field F. Let $G = \{g_1, \cdots, g_t\}$ be the reduced Gröbner basis of I with respect to the degree reverse lexicographic order. Then $g_1, \cdots, g_t$ are homogeneous polynomials in $\bar{F}[x_1, \cdots, x_n, z]$.

Let $E = B(n+1, d_1, \cdots, d_n)$ be the set of standard monomials with respect to I. Reducing g modulo G we obtain a polynomial which is a K-linear combination of all monomials of degree d in E with coefficients still algebraically independent over $\bar{F}$. Hence, from now on we assume that g is reduced modulo G and the coefficients of g are algebraically independent over $\bar{F}$.

In order to construct $B(n+1, d_1, \cdots, d_{n+1})$ from $B(n, d_1, \cdots, d_n)$ we need to compare $\mathrm{in}(I, g)$ and $\mathrm{in}(I)$. We recall here the incremental method to construct $\mathrm{in}(I, g)$ from $\mathrm{in}(I)$; for more details see [11, Section 3]. Let $B = B(n; d_1, \cdots, d_n)$ be the set of standard monomials with respect to $J = \pi(I)$. For every $i \geq 0$, denote by E_i the set of monomials of degree i in E. Note that, for $0 \leq i \leq \delta$, we have

$$E_i = B_i \cup zB_{i-1} \cup z^2 B_{i-2} \cup \cdots \cup z^{i-1} B_1 \cup z^i B_0,$$

and for $i > \delta$, we have $E_i = z^{i-\delta} E_\delta$.

Let $0 \leq i \leq \delta$, denote by $\mathbf{E}_i$ the column vector whose entries are the monomials in E_i listed in decreasing order with respect to the degree reverse lexicographic order . For each monomial $x^\alpha \in \mathbf{E}_i$, reducing the product $x^\alpha g \in R'_{i+d}$ modulo G we obtain a polynomial, say the reduced form of $x^\alpha g$, that is a K-linear combination of monomials in E_{i+d}. Note that each coefficient of the reduced form of $x^\alpha g$ is a $\bar{F}$-linear combination of coefficients of polynomial g. Let M_i denote the matrix such that

$$\mathbf{E}_i.g \equiv M_i \mathbf{E}_{i+d} \pmod{G}, \tag{3}$$

where $\mathbf{E}_{i+d}$ denotes the column vector whose entries are the monomials in E_{i+d} listed in decreasing order with respect to the degree reverse lexicographic order. Thus, each entry of matrix M_i is a $\bar{F}$-linear combination of coefficients of polynomial g. By [11, Lemma 3.2] the rows of M_i are linearly independent. This means that $\operatorname{rank}(M_i) = |E_i| = a_i + a_{i-1} + \cdots + a_0$. Furthermore, the monomials in $\mathbf{E}_{i+d}$ corresponding to the $|E_i|$ first linearly independent columns of M_i are the generators that will be added to $\operatorname{in}(I)$ to form $\operatorname{in}(I, g)$. Note that some of the monomials we add might be redundant. In this section, we will prove the following result which will be fundamental in our approach.

Theorem 2.6.5. *Let $(I, g) = (f_1, \cdots, f_n, g)$ be a generic homogeneous ideal of type $(n + 1; d_1, \cdots, d_n, d)$ in $R' = K[x_1, \cdots, x_n, z]$, where $d_1 \leq \cdots \leq d_n \leq d$. If $d \geq \sigma$ and Pardue's Conjecture is true for $J = \pi(I)$, then Pardue's Conjecture is also true for (I, g).*

The proof is technical and it needs a deep investigation given in Proposition 2.6.8, Proposition 2.6.11 and Proposition 2.6.12. Let us fix the property stated in Pardue's Conjecture.

Definition 2.6.6. Let I be a homogeneous ideal in $K[x_1, \cdots, x_n]$. Let x^α be a monomial in $K[x_1, \cdots, x_n]$ with $\max(x^\alpha) = m$ and $\deg(x^\alpha) = d$. We say x^α satisfies property P with respect to I if every monomial of degree d in the variables $x_1, \cdots, x_{m-1}$ is in the initial ideal of I.

Thus, Pardue's Conjecture is true for a generic homogeneous ideal I if and only if every minimal generator of $\operatorname{in}(I)$ satisfies property P with respect to I. First of all we notice that Theorem 2.6.5 can be deduced from [11, Proposition 3.12] when $d \geq \delta$. Indeed, by [11, Proposition 3.12], if $d \geq \delta$ then

$$\operatorname{in}(I, g) = (\operatorname{in}(I), z^{d-\delta} B_\delta, z^{d-\delta+2} B_{\delta-1}, \cdots, z^{\delta+d-2} B_1, z^{\delta+d} B_0).$$

Let x^μ be a generator of $\operatorname{in}(I, g)$ in $z^{d-\delta} B_\delta, z^{d-\delta+2} B_{\delta-1}, \cdots, z^{\delta+d-2} B_1, z^{\delta+d} B_0$. We claim x^μ satisfies property P with respect to (I, g). Indeed, if $d > \delta$ then x^μ is divisible by z and $\deg(x^\mu) = k > \delta$. Hence, every monomial x^α of degree k in variables $x_1, \cdots, x_n$ is not in $E_k = z^{k-\delta} E_\delta$, so that $x^\alpha \in \operatorname{in}(I) \subset \operatorname{in}(I, g)$. If $d = \delta$ then

$$\operatorname{in}(I, g) = (\operatorname{in}(I), x_n^\delta, z^2 B_{\delta-1}, \cdots, z^{2\delta-2} B_1, z^{2\delta} B_0).$$

If x^μ is a monomial in $z^2 B_{\delta-1}, \cdots, z^{2\delta-2} B_1, z^{2\delta} B_0$, then x^μ satisfies property P with respect to (I, g) through an analogous argument as the case $d > \delta$. On the other hand, it is not hard to see that x_n^δ also satisfies property P with respect to (I, g). Thus, if Pardue's Conjecture is true for J with $d \geq \delta$, then every minimal generator of $\operatorname{in}(I, g)$ satisfies property P with respect to (I, g). This means Pardue's Conjecture is true for (I, g).

Consider now the case $d < \delta$. Set $i^* = \lfloor \frac{\delta - d}{2} \rfloor$. The following lemma will be useful for proving Proposition 2.6.8.

Lemma 2.6.7. *For $i > j \geq i^*$, we have $a_{d+i} \leq a_{d+j}$. Furthermore, the monomials of B_{d+i} are multiples of the a_{d+i} smallest monomials in B_{d+j} with respect to the degree reverse lexicographic order.*

Proof. Since $d + i^* \geq \lfloor \frac{\delta}{2} \rfloor \geq \sigma$, so $d + i > d + j \geq \sigma$. Hence, $a_{d+i} \leq a_{d+j}$. Let S denote the subset of B_{d+j} consisting of the a_{d+i} smallest monomials in B_{d+j} with respect to the degree reverse lexicographic order. By [11, Lemma 3.5 (ii)] we have $B_{d+i} = x_n^{i-j} S$. $\square$

By convention, we use the following notation. Let B be a finite subset of monomials in $R = K[x_1, \cdots, x_n]$ and denote by $\mathbf{B} = \{x^{\alpha_1}, x^{\alpha_2}, \cdots, x^{\alpha_m}\}$ the set of monomials in B listed in decreasing order with respect to the degree reverse lexicographic order. Let S be a subset of $\{1, 2, \cdots, m\}$ and denote $\mathbf{B}^S = \{x^{\alpha_i} \in B \mid i \in S\}$. The set of generators of $\text{in}(I, g)$ can be described as the following.

Proposition 2.6.8. *Let $(I, g) = (f_1, \cdots, f_n, g)$ be a generic ideal of type $(n+1; d_1, \cdots, d_n, d)$ in $R' = K[x_1, \cdots, x_n, z]$, where $d_1 \leq \cdots \leq d_n \leq d$ and $d < \delta$. Let $B = B(n; d_1, \cdots, d_n)$ be the set of standard monomials with respect to $J = \pi(I)$.*

(1) *If $\delta - d = 2k$, where k is a positive integer, then*

$$\text{in}(I, g) = (\text{in}(I), \mathbf{B}_d^{\{1\}}, \mathbf{B}_{d+1}^{S_1}, \mathbf{B}_{d+2}^{S_2}, \cdots, \mathbf{B}_{d+k}^{S_k}, z^2 B_{d+k-1}, z^4 B_{d+k-2}, \cdots, z^{2(d+k)} B_0),$$

(2) *If $\delta - d = 2k + 1$, where k is a positive integer, then*

$$\text{in}(I, g) = (\text{in}(I), \mathbf{B}_d^{\{1\}}, \mathbf{B}_{d+1}^{S_1}, \mathbf{B}_{d+2}^{S_2}, \cdots, \mathbf{B}_{d+k}^{S_k}, B_{d+k+1}, z B_{d+k}, z^3 B_{d+k-1}, \cdots, z^{2(d+k)+1} B_0),$$

(3) *If $\delta - d = 1$ then* $\text{in}(I, g) = (\text{in}(I), \mathbf{B}_d^{\{1\}}, B_{d+1}, z B_d, z^3 B_{d-1}, \cdots, z^{2d+1} B_0)$,

where in (1) and (2), S_i is a subset of $\{1, 2, \cdots, a_{d+i}\}$ containing a_i elements, for every $i = 1, 2, \cdots, k$.

Proof. Since g is a combination of all monomials in $E_d = B_d \cup z B_{d-1} \cup \cdots \cup z^d B_0$, we can write

$$g = \mathbf{v}_d \mathbf{B}_d + \mathbf{v}_{d-1} \mathbf{B}_{d-1} z + \cdots + \mathbf{v}_1 \mathbf{B}_1 z^{d-1} + \mathbf{v}_0 z^d,$$

where $\mathbf{v}_i$ is the row vector of the coefficients of g corresponding to the monomials in $\mathbf{B}_i$. Denote the last coefficient of $\mathbf{v_d}$ by c_d. Note that c_d is the coefficient corresponding to the monomial x_n^d. Set $\mathbf{v}_d^* = \mathbf{v}_d \setminus \{c_d\}$. We will construct a set of generators for $\text{in}(I, g)$ by using incremental method. According to equation (3), $\mathbf{E}_i . g \equiv M_i \mathbf{E}_{i+d} \pmod{G}$, for each i from 0 to δ, we find the monomials that will be added to $\text{in}(I)$.

• For $i = 0$, we have $E_0 = \{1\}$ and M_0 is a row matrix $\begin{pmatrix} \mathbf{v}_d & \mathbf{v}_{d-1} & \cdots & \mathbf{v}_1 & \mathbf{v}_0 \end{pmatrix}$. Hence, the first column of M_0 is linearly independent. Thus, the largest monomial of $\mathbf{B}_d$ will be added to $\text{in}(I)$.

• For $1 \leq i \leq i^* = \lfloor \frac{\delta - d}{2} \rfloor = k$ (in the case $\delta - d \geq 2$), we have

$$E_i = B_i \cup z B_{i-1} \cup \cdots \cup z^{i-1} B_1 \cup z^i B_0,$$

and

$$E_{d+i} = B_{d+i} \cup z B_{d+i-1} \cup \cdots \cup z^{i-1} B_{d+1} \cup z^i B_d \cup \cdots \cup z^{d+i} B_0.$$

Therefore equation (3) can be explicitly written in the following form

$$
(4)\qquad \begin{pmatrix} \mathbf{B}_i \\ z\mathbf{B}_{i-1} \\ \vdots \\ z^{i-1}\mathbf{B}_1 \\ z^i\mathbf{B}_0 \end{pmatrix} .g \equiv \begin{array}{c} \begin{array}{ccccccc} \mathbf{B}_{d+i} & z\mathbf{B}_{d+i-1} & \cdots & z^{i-1}\mathbf{B}_{d+1} & z^i\mathbf{B}_d & \cdots & z^{d+i} \end{array} \\ \begin{pmatrix} \Gamma_{i,d+i} & \Gamma_{i,d+i-1} & \cdots & \Gamma_{i,d+1} & \Gamma_{i,d} & \cdots & \Gamma_{i,0} \\ 0 & \Gamma_{i-1,d+i-1} & \cdots & \Gamma_{i-1,d+1} & \Gamma_{i-1,d} & \cdots & \Gamma_{i-1,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{1,d+1} & \Gamma_{1,d} & \cdots & \Gamma_{1,0} \\ 0 & 0 & \cdots & 0 & \Gamma_{0,d} & \cdots & \Gamma_{0,0} \end{pmatrix} \end{array},
$$

where the entries of block $\Gamma_{j,l}$, for $0 \leq j \leq i$ and $0 \leq l \leq d+i$, are the coefficients corresponding to the monomials in $z^{d+i-l}B_l$, in the reduced form of the polynomials in $z^{i-j}gB_j$.

Denote by $A_i, A_{i-1}, \cdots, A_1, A_0$ the submatrices of M_i formed by the columns corresponding to the monomials in $\mathbf{B}_{d+i}, z\mathbf{B}_{d+i-1}, \cdots, z^{i-1}\mathbf{B_{d+1}}, z^i\mathbf{B_d}$ respectively.

We now consider the block $\Gamma_{i,d+i}$. Note that the entries of $\Gamma_{i,d+i}$ are the $\bar{F}$-linear combinations of the coefficients in $\mathbf{v}_d$. Since $i \leq \dfrac{\delta - d}{2}$, we have $i < \sigma$ and

$$
\mathbf{B}_i = \widetilde{\mathbf{B}}_i \cup x_n\widetilde{\mathbf{B}}_{i-1} \cup \cdots \cup x_n^{i-1}\widetilde{\mathbf{B}}_1 \cup \{x_n^i\}.
$$

Since $d+i \leq \delta - i$, we have $a_{d+i} \geq a_i$ and the a_i smallest monomials in $\mathbf{B}_{d+i}$ are

$$
x_n^d\mathbf{B}_i = x_n^d\widetilde{\mathbf{B}}_i \cup x_n^{d+1}\widetilde{\mathbf{B}}_{i-1} \cup \cdots \cup x_n^{d+i-1}\widetilde{\mathbf{B}}_1 \cup \{x_n^{d+i}\}.
$$

Let $x^\alpha \in \mathbf{B}_i$. Suppose $x^\alpha = x_n^j x^\beta \in x_n^j\widetilde{\mathbf{B}}_{i-j}$, for some $0 \leq j \leq i$ and $x^\beta \in \widetilde{\mathbf{B}}_{i-j}$. Then

$$
x^\alpha x_n^d = x_n^{d+j}x^\beta \in x_n^{d+j}\widetilde{\mathbf{B}}_{i-j} \subset \mathbf{B}_{d+i}.
$$

Thus, the term $c_d.x^\alpha x_n^d$ of the product $x^\alpha g$ is reduced mod G. Therefore, in the coefficients of the reduced form of the product $x^\alpha.g$, c_d will appear only in the coefficient of the monomial $x_n^{d+j}x^\beta \in \mathbf{B}_{d+i}$. Thus,

$$
\Gamma_{i,d+i} = \begin{pmatrix} L_{1,1} & \cdots & c_d + L_{1,s} & L_{1,s+1} & \cdots & L_{1,a_{d+i}} \\ L_{2,1} & \cdots & L_{2,s} & c_d + L_{2,s+1} & \cdots & L_{2,a_{d+i}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{a_i,1} & \cdots & L_{a_i,s} & L_{a_i,s+1} & \cdots & c_d + L_{a_i,a_{d+i}} \end{pmatrix},
$$

where $s = a_{d+i} - a_i + 1$ and $L_{a,b}$, for $1 \leq a \leq a_i$ and $1 \leq b \leq a_{d+i}$, is a $\bar{F}$-linear combination of the coefficients in $\mathbf{v}_d^*$. Hence, the a_i last columns of $\Gamma_{i,d+i}$ are linearly independent. So, $\operatorname{rank}(\Gamma_{i,d+i}) = a_i$. This implies that the a_i first linearly independent columns of M_i are the a_i first linearly independent columns of A_i (previously defined). Define S_i to be the subset of $\{1, 2, \cdots, a_{d+i}\}$ such that its elements are the indices of the a_i first linearly independent columns of A_i. Then the monomials in $\mathbf{B}_{d+i}^{S_i}$ will be added to in(I). Since the a_{i-1} last columns of $\Gamma_{i-1,d+i-1}$ are linearly independent again, we have

$$
\operatorname{rank}\begin{pmatrix} \Gamma_{i,d+i} & \Gamma_{i,d+i-1} \\ 0 & \Gamma_{i-1,d+i-1} \end{pmatrix} = a_i + a_{i-1}.
$$

Therefore, the a_{i-1} next linearly independent columns of M_i are the a_{i-1} first linearly independent columns of A_{i-1} and so on. We have the $a_i + a_{i-1} + \cdots + a_0$ first linearly independent columns of M_i are the $a_i, a_{i-1}, \cdots, a_1, a_0$ first linearly independent columns of $A_i, A_{i-1}, \cdots, A_1, A_0$ respectively.

However, the monomials in $z\mathbf{B}_{d+i-1}, \cdots, z^{i-1}\mathbf{B}_{d+1}, z^i\mathbf{B}_d$ corresponding to the first linearly independent columns of $A_{i-1}, \cdots, A_1, A_0$ respectively are redundant since they are multiples of the monomials that were already added to $\mathrm{in}(I)$ in the steps $i-1, \cdots, 1, 0$. Thus, in the step i, only the monomials in $\mathbf{B}_{d+i}^{S_i}$ will be added to $\mathrm{in}(I)$.

Moreover, in the case $\delta - d = 2k$ we have $\mathbf{B}_{d+k}^{S_k} = B_{d+k}$. Indeed, since $d + k = \delta - k$, by [11, Lemma 3.4] one has $a_{d+k} = a_{\delta-k} = a_k$. Hence, $S_k = \{1, 2, \cdots, a_{d+k}\}$.

• For $i^* < i < \delta - d$ (in the case $\delta - d \geq 3$), equation (3) also has form as in (4). Let Λ_i denote the square submatrix of M_i given by

$$\Lambda_i = \begin{array}{c} \begin{array}{cccc} \mathbf{B}_{d+i} & z\mathbf{B}_{d+i-1} & \cdots & z^{d+2i-\delta}\mathbf{B}_{\delta-i} \end{array} \\ \begin{pmatrix} \Gamma_{i,d+i} & \Gamma_{i,d+i-1} & \cdots & \Gamma_{i,\delta-i} \\ 0 & \Gamma_{i-1,d+i-1} & \cdots & \Gamma_{i-1,\delta-i} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_{\delta-d-i,\delta-i} \end{pmatrix} \end{array}.$$

Then, M_i has form

$$M_i = \begin{pmatrix} \Lambda_i & \Omega \\ 0 & M_{\delta-d-i-1} \end{pmatrix}.$$

By [11, Proposition 3.16] Λ_i is nonsingular. Hence, the first linearly independent columns of M_i are given by all the columns of Λ_i and the columns corresponding to the first linearly independent columns of $M_{\delta-d-i-1}$. Note that the monomials in E_{d+i} corresponding to the first linearly independent columns of $M_{\delta-d-i-1}$ are redundant since they are multiplies of monomials were already added to $\mathrm{in}(I)$ in the step $\delta - d - i - 1$. Furthermore, by using Lemma 2.6.7, for $i = i^* + 1, i^* + 2, \cdots, \delta - d - 1$, we obtain the following.

If $\delta - d = 2k$, where k is an integer and $k \geq 2$, then the monomials in $z^2B_{d+k-1}, z^4B_{d+k-2}, \cdots, z^{2k-2}B_{d+1}$ will be added to $\mathrm{in}(I)$.

If $\delta - d = 2k + 1$, where k is a positive integer, then the monomials in $B_{d+k+1}, zB_{d+k}, z^3B_{d+k-1}, \cdots, z^{2k-1}B_{d+1}$ will be added to $\mathrm{in}(I)$.

• For $\delta - d \leq i \leq \delta$, by [11, Corollary 3.11] the $|E_i|$ first columns of M_i are linearly independent. Hence, we obtain the following.

If $\delta - d \geq 2$ then the monomials in $z^{\delta-d}B_d$, $z^{\delta-d+2}B_{d-1}$, $\cdots$, $z^{\delta+d-2}B_1$, $z^{\delta+d}B_0$ will be added to $\mathrm{in}(I)$.

If $\delta - d = 1$ then the monomials in B_{d+1}, zB_d, z^3B_{d-1}, $\cdots$, $z^{2d+1}B_0$ will be added to $\mathrm{in}(I)$. □

Remark 2.6.9. The set of generators of $\mathrm{in}(I, g)$, which appears in Proposition 2.6.8, is not minimal. For instance the monomial $z^{\delta-d}\mathbf{B}_d^{\{1\}}$ is a multiple of $\mathbf{B}_d^{\{1\}}$.

The following lemma will be useful for proving Proposition 2.6.11.

Lemma 2.6.10. *In the case $\delta - d \geq 2$ we have $B_{d+i} \subset \text{in}(I, g)$ for every $i > i^*$.*

Proof. If $\delta - d = 2k$, where k is a positive integer, then $i^* = k$. By Proposition 2.6.8, we have $B_{d+i^*} = \mathbf{B}_{d+i^*}^{S_{i^*}} \subset \text{in}(I, g)$. Hence, by Lemma 2.6.7, $B_{d+i} \subset \text{in}(I, g)$ for every $i > i^*$.

If $\delta - d = 2k + 1$, where k is a positive integer, then $i^* = k$. By Proposition 2.6.8, we have $B_{d+i^*+1} \subset \text{in}(I, g)$. Hence, by Lemma 2.6.7, $B_{d+i} \subset \text{in}(I, g)$ for every $i > i^* + 1$. Thus, $B_{d+i} \subset \text{in}(I, g)$ for every $i > i^*$. □

In the next proposition, we will prove that the generators of $\text{in}(I, g)$ not in $\text{in}(I)$, $\mathbf{B}_d^{\{1\}}$, $\mathbf{B}_{d+1}^{S_1}, \cdots, \mathbf{B}_{d+i^*}^{S_{i^*}}$ satisfy property P with respect to (I, g) in any case.

Proposition 2.6.11. (1) *If $\delta - d = 2k$, where k is a positive integer, then the generators of* $\text{in}(I, g)$ *in $z^2 B_{d+k-1}$, $z^4 B_{d+k-2}$, $\cdots$, $z^{2(d+k)} B_0$ satisfy property P with respect to (I, g).*

(2) *If $\delta - d = 2k + 1$, where k is a non-negative integer, then the generators of* $\text{in}(I, g)$ *in B_{d+k+1}, zB_{d+k}, $z^3 B_{d+k-1}$, $\cdots$, $z^{2(d+k)+1} B_0$ satisfy property P with respect to (I, g).*

Proof. (1) If x^μ is a generator of $\text{in}(I, g)$ in $z^2 B_{d+k-1}$, $z^4 B_{d+k-2}$, $\cdots$, $z^{2(d+k)} B_0$, then x^μ is divisible by z and $\deg(x^\mu) = l > d + k$. For every monomial x^α of degree l in variables $x_1, \cdots, x_n$, if $x^\alpha \in B_l$, by Lemma 2.6.10, then we have $x^\alpha \in \text{in}(I, g)$. Otherwise $x^\alpha \notin B_l$, so that $x^\alpha \in \text{in}(I) \subset \text{in}(I, g)$. Thus, x^μ satisfies property P with respect to (I, g).

(2) If x^μ is a generator of $\text{in}(I, g)$ in B_{d+k+1}, then x^μ is divisible by x_n. Indeed, by [11, Lemma 3.4], $B_{d+k+1} = B_{\delta-k} = x_n^{d+1} B_k$. Hence, for every monomial x^α of degree $d + k + 1$ in variables $x_1, \cdots, x_{n-1}$, we have $x^\alpha \notin B_{d+k+1}$, so that $x^\alpha \in \text{in}(I) \subset \text{in}(I, g)$. Thus, x^μ satisfies property P with respect to (I, g).

If x^μ is a generator of $\text{in}(I, g)$ in zB_{d+k}, $z^3 B_{d+k-1}$, $\cdots$, $z^{2(d+k)+1} B_0$. Then, by an argument as in (1), x^μ satisfies property P with respect to (I, g). □

We still have to prove that the minimal generators of $\text{in}(I, g)$ in $\mathbf{B}_{d+1}^{S_1}, \cdots, \mathbf{B}_{d+i^*}^{S_{i^*}}$ satisfy property P with respect to (I, g). Under condition $d \geq \sigma$, we get the following.

Proposition 2.6.12. *If $\sigma \leq d \leq \delta - 2$, then the generators of* $\text{in}(I, g)$ *in $\mathbf{B}_{d+1}^{S_1}, \cdots, \mathbf{B}_{d+i^*}^{S_{i^*}}$ satisfy property P with respect to (I, g).*

Proof. Since $d \geq \sigma$, by Propsition 2.6.2 the monomials in $B_{d+1}, \cdots, B_{d+i^*}$ are divisible by x_n. Hence, if x^μ is a generator of $\text{in}(I, g)$ in $\mathbf{B}_{d+i}^{S_i}$, for some $1 \leq i \leq i^*$, then x^μ is divisible by x_n. This implies x^μ satisfies property P with respect to (I, g) because for every monomial x^α of degree $d+i$ in variables $x_1, \cdots, x_{n-1}$, we have $x^\alpha \notin B_{d+i}$, so that $x^\alpha \in \text{in}(I) \subset \text{in}(I, g)$. □

Proof of Theorem 2.6.5. If $\delta - d = 2k$, where k is a positive integer, then by Proposition 2.6.8 we have

$$\text{in}(I, g) = (\text{in}(I), \mathbf{B}_d^{\{1\}}, \mathbf{B}_{d+1}^{S_1}, \cdots, \mathbf{B}_{d+k}^{S_k}, z^2 B_{d+k-1}, z^4 B_{d+k-2}, \cdots, z^{2(d+k)} B_0).$$

The monomial $\mathbf{B}_d^{\{1\}}$ satisfies property P with respect to (I, g) because it is the largest monomial of $\mathbf{B}_d$. By Proposition 2.6.11 and Proposition 2.6.12 the generators of $\text{in}(I, g)$ in $z^2 B_{d+k-1}, z^4 B_{d+k-2}, \cdots, z^{2(d+k)} B_0$ and in $\mathbf{B}_{d+1}^{S_1}, \cdots, \mathbf{B}_{d+k}^{S_k}$ satisfy property P with respect to (I, g). Hence, if Pardue's Conjecture is true for $J = \pi(I)$, then every minimal generator of $\text{in}(I, g)$ satisfies property P with respect to (I, g), so that Pardue's Conjecture is true for $\text{in}(I, g)$.

In case $\delta - d = 2k+1$, where k is a non-negative integer, theorem is proved by a completely analogous argument as above. $\square$

From Proposition 2.6.8, we have the following corollary which describes more explicitly the set of the standard monomials with respect to (I, g) in case $d < \delta$.

Corollary 2.6.13. *Let $(I, g) = (f_1, \cdots, f_n, g)$ be a generic homogeneous ideal of type $(n+1; d_1, \cdots, d_n, d)$ in $K[x_1, \cdots, x_n, z]$, where $d_1 \leq \cdots \leq d_n \leq d$ and $d < \delta$. Let $B = B(n; d_1, \cdots, d_n)$ and $F = B(n+1; d_1, \cdots, d_n, d)$.*

(1) *If $\delta - d = 2k$, where k is a positive integer, then*

$F_0 = B_0$,

$F_i = B_i \cup zF_{i-1}$ *for every* $1 \leq i \leq d-1$,

$F_d = \mathbf{B}_d^{\{2,\cdots,a_d\}} \cup zF_{d-1}$,

$F_{d+i} = \mathbf{B}_{d+i}^{\{1,2,\cdots,a_{d+i}\}\setminus S_i} \cup zF_{d+i-1}$ *for every* $1 \leq i \leq k-1$,

$F_{d+k} = zF_{d+k-1}$, $F_{d+k+1} = z^3 F_{d+k-2}, \cdots, F_{2(d+k)-1} = z^{2(d+k)-1} F_0$.

(2) *If $\delta - d = 2k+1$, where k is a positive integer, then*

$F_0 = B_0$,

$F_i = B_i \cup zF_{i-1}$ *for every* $1 \leq i \leq d-1$,

$F_d = \mathbf{B}_d^{\{2,\cdots,a_d\}} \cup zF_{d-1}$,

$F_{d+i} = \mathbf{B}_{d+i}^{\{1,2,\cdots,a_{d+i}\}\setminus S_i} \cup zF_{d+i-1}$ *for every* $1 \leq i \leq k$,

$F_{d+k+1} = z^2 F_{d+k-1}$, $F_{d+k+2} = z^4 F_{d+k-2}, \cdots, F_{2(d+k)} = z^{2(d+k)} F_0$.

Thus, in order to construct $F = B(n+1, d_1, \cdots, d_n, d)$ from $B = B(n; d_1, \cdots, d_n)$ we only need to know explicitly the monomials in $\mathbf{B}_{d+1}^{S_1}, \cdots, \mathbf{B}_{d+i^*}^{S_{i^*}}$.

In the following example, we construct $\text{in}(I, g)$ from $\text{in}(I)$ according to Proposition 2.6.8. Moreover, we construct $F = B(n+1; d_1, \cdots, d_n, d)$ from $B(n; d_1, \cdots, d_n)$ according to Corollary 2.6.13.

Example 2.6.14. Let $(I, g) = (f_1, \cdots, f_4, g)$ be the generic ideal of type $(5; 2, 3, 3, 4, 5)$ in $K[x_1, \cdots, x_4, z]$. Let $B = B(4; 2, 3, 3, 4)$ as in Example 2.6.4. Then $\delta = 8, \sigma = 4, d = 5$ and $i^* = \lfloor \frac{\delta - d}{2} \rfloor = 1$. We write g in reduced form as the following

$$g = \mathbf{v}_5 \mathbf{B}_5 + \mathbf{v}_4 \mathbf{B}_4 z + \cdots + \mathbf{v}_1 \mathbf{B}_1 z^4 + z^5.$$

We will construct a set of generators for $\text{in}(I, g)$ by using incremental method as in the proof Proposition 2.6.8. According to equation (3), $\mathbf{E}_i.g \equiv M_i \mathbf{E}_{i+d} \pmod G$, for each i from 0 to 8, we find the monomials that will be added to $\text{in}(I)$.

For $i = 0$, the largest monomials of $\mathbf{B}_5$ will be added to $\text{in}(I)$, in this case it is $x_1 x_2 x_3^2 x_4^2$.

For $i = 1$,

$$\begin{array}{cc} & \begin{array}{ccccc} \mathbf{B}_6 & z\mathbf{B}_5 & \cdots & z^5\mathbf{B}_1 & z^6 \end{array} \\ \mathbf{E}_1.g \equiv M_1\mathbf{E}_6 \;\Leftrightarrow\; \begin{pmatrix} \mathbf{B}_1 \\ z \end{pmatrix}.g \equiv & \begin{pmatrix} \Gamma_{1,6} & \Gamma_{1,5} & \cdots & \Gamma_{1,1} & \Gamma_{1,0} \\ 0 & \Gamma_{0,5} & \cdots & \Gamma_{0,1} & \Gamma_{0,0} \end{pmatrix}, \end{array}$$

where $\mathbf{B}_1 = \widetilde{\mathbf{B}}_1 \cup \{x_4\}$ and $\mathbf{B}_6 = x_4^4\widetilde{\mathbf{B}}_2 \cup x_4^5\widetilde{\mathbf{B}}_1 \cup \{x_4^6\}$. The monomials in $\mathbf{B}_6$ corresponding to the $a_1 = 4$ first linearly independent columns of $\Gamma_{1,6}$ will be added to $\text{in}(I)$. By using Macaulay2 to compute $\text{in}(I, g)$, we see that the 4 largest monomials of $\mathbf{B}_6$ are the minimal generators of $\text{in}(I, g)$. This means that the 4 first columns of $\Gamma_{1,6}$ are linearly independent, so that $\mathbf{B}_6^{S_1} = \mathbf{B}_6^{\{1,2,3,4\}}$.

For $2 \le i \le 8$ the monomials will be added to $\text{in}(I)$ are B_7, zB_6, z^3B_5, z^5B_4, z^7B_3, z^9B_2, $z^{11}B_1$, z^{13}. Thus, the set of generators of $\text{in}(I, g)$ is

$$\text{in}(I, g) = (\text{in}(I), \mathbf{B}_5^{\{1\}}, \mathbf{B}_6^{\{1,2,3,4\}}, B_7, zB_6, z^3B_5, z^5B_4, z^7B_3, z^9B_2, z^{11}B_1, z^{13}).$$

Let $F = B(5; 2, 3, 3, 4, 5)$ be the set of the standard monomials with respect to (I, g). Denote by $f_i = |F_i|$ and $f_i' = |\widetilde{F}_i|$. By Corollary 2.6.13, we have

$$\begin{array}{ll} F_0 = \{1\} & f_0 = 1. \\ F_1 = B_1 \cup \{z\} & f_1' = 4, f_1 = 5. \\ F_2 = B_2 \cup zF_1 & f_2' = 9, f_2 = 14. \\ F_3 = B_3 \cup zF_2 & f_3' = 14, f_3 = 28. \\ F_4 = B_4 \cup zF_3 & f_4' = 16, f_4 = 44. \\ F_5 = \widetilde{F}_5 \cup zF_4 \text{ where } \widetilde{F}_5 = \mathbf{B}_5^{\{2,3,\cdots,14\}} & f_5' = 13, f_5 = 57. \\ F_6 = \widetilde{F}_6 \cup zF_5 \text{ where } \widetilde{F}_6 = \mathbf{B}_6^{\{5,6,\cdots,9\}} & f_6' = 5, f_6 = 62. \\ F_7 = z^2F_5;\; F_8 = z^4F_4;\; \cdots; F_{11} = z^{10}F_1;\; F_{12} = z^{12}F_0. & \end{array}$$

In [11, Conjecture 3.14], it is conjectured that $\mathbf{B}_{d+i}^{S_i}$ are the a_i largest monomials of $\mathbf{B}_{d+i}$ for every $i = 0, \cdots, i^*$. However, in the following example, we show that this conjecture is not true.

Example 2.6.15. Let $(I, g) = (f_1, \cdots, f_5, g)$ be the generic ideal of type $(6; 2, 3, 3, 4, 5, 5)$ in $K[x_1, \cdots, x_5, z]$. Let $F = B(5; 2, 3, 3, 4, 5)$ as in Example 2.6.14 with the variable x_5 instead of variable z. Then $\delta = 12, \sigma = 6$, $d = 5$ and $i^* = \lfloor \frac{\delta - d}{2} \rfloor = 3$. Here F plays the same role of B in Proposition 2.6.8. We write g in reduced form as the following

$$g = \mathbf{v}_5\mathbf{F}_5 + \mathbf{v}_4\mathbf{F}_4z + \cdots + \mathbf{v}_1\mathbf{F}_1z^4 + z^5.$$

According to the incremental method, for $i = 0$, the largest monomials of $\mathbf{F}_5$ will be added to $\text{in}(I)$, in this case it is $x_2^2x_3x_4^2$.

For $i = 1$,

$$\begin{array}{cc} & \begin{array}{ccccc} \mathbf{F}_6 & z\mathbf{F}_5 & \cdots & z^5\mathbf{F}_1 & z^6 \end{array} \\ \mathbf{E}_1.g \equiv M_1\mathbf{E}_6 \;\Leftrightarrow\; \begin{pmatrix} \mathbf{F}_1 \\ z \end{pmatrix}.g \equiv & \begin{pmatrix} \Gamma_{1,6} & \Gamma_{1,5} & \cdots & \Gamma_{1,1} & \Gamma_{1,0} \\ 0 & \Gamma_{0,5} & \cdots & \Gamma_{0,1} & \Gamma_{0,0} \end{pmatrix}, \end{array}$$

where $\mathbf{F}_1 = \widetilde{\mathbf{F}}_1 \cup \{x_5\}$ and $\mathbf{F}_6 = \widetilde{\mathbf{F}}_6 \cup x_5\widetilde{\mathbf{F}}_5 \cdots \cup x_5^5\widetilde{\mathbf{F}}_1 \cup \{x_5^6\}$. The monomials in $\mathbf{F}_6$ corresponding to the $f_1 = 5$ first linearly independent columns of $\Gamma_{1,6}$ will be added to $\mathrm{in}(I)$. We have

$$\mathbf{F}_1.g \longleftrightarrow \Gamma_{1,6}\mathbf{F}_6 \;\Leftrightarrow\; \begin{pmatrix}\widetilde{\mathbf{F}}_1\\ x_5\end{pmatrix}.g \longleftrightarrow \begin{array}{c} \begin{array}{ccccc} \widetilde{\mathbf{F}}_6 & x_5\widetilde{\mathbf{F}}_5 & \cdots & x_5^5\widetilde{\mathbf{F}}_1 & x_5^6 \end{array} \\ \begin{pmatrix} \Omega_{1,6} & \Omega_{1,5} & \cdots & \Omega_{1,1} & \Omega_{1,0}\\ 0 & \Omega_{0,5} & \cdots & \Omega_{0,1} & \Omega_{0,0}\end{pmatrix} \end{array},$$

Since $|\widetilde{\mathbf{F}}_1| = f_1' = 4$ and $|\widetilde{\mathbf{F}}_6| = f_6' = 5$, we get $\mathrm{rank}(\Omega_{1,6}) \leq 4$. By using Macaulay2 to compute $\mathrm{in}(I, g)$, we see that the 4 largest monomials of $\mathbf{F}_6$ are the minimal generators of $\mathrm{in}(I, g)$. This means that the 4 first columns of $\Omega_{1,6}$ are linearly independent. Hence the 5 first linearly independent columns of $\Gamma_{1,6}$ are the 4 first columns and the 6-th column. Thus $\mathbf{F}_6^{S_1} = \mathbf{F}_6^{\{1,2,3,4\}} \cup \mathbf{F}_6^{\{6\}}$. However the monomial $\mathbf{F}_6^{\{6\}}$ is not a minimal generator because $\mathbf{F}_6^{\{6\}} = x_5\mathbf{F}_5^{\{1\}}$ and $\mathbf{F}_5^{\{1\}}$ was already added to $\mathrm{in}(I)$ in step $i = 0$.

2.7. **Application to Pardue and Fröberg's Conjectures.** In [40, Theorem 2], Pardue proved that Fröberg's Conjecture is equivalent to Pardue's Conjecture. In order to prove the equivalence of the conjectures, Pardue used the notion of semi-regular sequences that was introduced in [40, Section 3]. Regular sequences and semi-regular sequences can be characterized by Hilbert series.

Proposition 2.7.1. [40, Proposition 1] *Let $A = K[x_1, \cdots, x_n]/I$, where I is a homogeneous ideal, and $f_1, \cdots, f_r$ are homogeneous polynomials of degree $d_1, \cdots, d_r$. Then,*

(1) *$f_1, \cdots, f_r$ is a semi-regular sequence on A if and only if for all $s = 1, \cdots, r$*
$$P_{A/(f_1,\cdots,f_s)}(z) = \left\lceil \left(\textstyle\prod_{i=1}^s (1 - z^{d_i})\right) P_A(z) \right\rceil.$$
(2) *$f_1, \cdots, f_r$ is a regular sequence on A if and only if*
$$P_{A/(f_1,\cdots,f_r)}(z) = \left(\textstyle\prod_{i=1}^r (1 - z^{d_i})\right) P_A(z).$$

In [40, Theorem 2], Pardue also proved that Pardue's Conjecture is equivalent to the following conjecture.

Conjecture 2.7.2. [40, Conjecture C] *Let $I = (f_1, \cdots, f_n)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $K[x_1, \cdots, x_n]$. Then $x_n, x_{n-1}, \cdots, x_1$ is a semi-regular sequence on $A = K[x_1, \cdots, x_n]/I$.*

We apply now Theorem 2.6.5 to get partial answers to Pardue's Conjecture and Conjecture 2.7.2. Let $d_1 \leq \cdots \leq d_n$ be n positive integers. For every $1 \leq i \leq n$, we set

$$\delta_i = d_1 + \cdots + d_i - i,$$

$$\sigma_i = \min\left\{\delta_{i-1}, \left\lfloor \frac{\delta_i}{2} \right\rfloor\right\} \text{ for all } i \geq 2.$$

Theorem 2.7.3. *Let $I = (f_1, \cdots, f_n)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $K[x_1, \cdots, x_n]$ with $n \leq 3$ and $d_1 \leq \cdots \leq d_n$. Then, Pardue's Conjecture is true for I.*

Proof. It is known that Pardue's Conjecture is true in case $n \leq 2$. For $n = 3$, we have $J = \pi(I)$ is a generic ideal of type $(2; d_1, d_2)$. Hence, Pardue's Conjecture is true for J. Since

$$d_3 \geq \sigma_2 = \min\left\{d_1 - 1, \left\lfloor \frac{d_1 + d_2 - 2}{2} \right\rfloor\right\},$$

by Theorem 2.6.5 we have that Pardue's Conjecture is true for I. $\square$

Theorem 2.7.4. *Let $I = (f_1, \cdots, f_n)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $K[x_1, \cdots, x_n]$ with $n \geq 4$ and $d_1 \leq \cdots \leq d_n$. If $d_i \geq \sigma_{i-1}$ for all $4 \leq i \leq n$, then Pardue's Conjecture is true for I.*

Proof. We prove by induction on n. For $n = 4$, we have $J = \pi(f_1, f_2, f_3)$ is a generic ideal of type $(3; d_1, d_2, d_3)$. By Theorem 2.7.3, Pardue's Conjecture is true for J. Since $d_4 \geq \sigma_3$, by Theorem 2.6.5 we have that Pardue's Conjecture is true for I.

For $n > 4$, we have $J = \pi(f_1, \cdots, f_{n-1})$ is a generic ideal of type $(n-1; d_1, \cdots, d_{n-1})$ with $d_i \geq \sigma_{i-1}$ for all $4 \leq i \leq n-1$. Hence, by induction Pardue's Conjecture is true for J. Since, $d_n \geq \sigma_{n-1}$, by Theorem 2.6.5 we have that Pardue's Conjecture is true for I. $\square$

Since Pardue's Conjecture is equivalent to Conjecture 2.7.2, we also get a partial answer to Conjecture 2.7.2.

Corollary 2.7.5. *Let $I = (f_1, \cdots, f_n)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_n)$ in $K[x_1, \cdots, x_n]$ with $d_1 \leq \cdots \leq d_n$. If $n \leq 3$ or $n \geq 4$ and $d_i \geq \sigma_{i-1}$ for all $4 \leq i \leq n$, then $x_n, x_{n-1}, \cdots, x_1$ is a semi-regular sequence on $K[x_1, \cdots, x_n]/I$.*

We apply now above results to prove a new partial answer for Fröberg's Conjecture.

Theorem 2.7.6. *Let $I = (f_1, \cdots, f_r)$ be a generic homogeneous ideal of type $(n; d_1, \cdots, d_r)$ in $R = K[x_1, \cdots, x_n]$ with $r \leq n + 2$ and $d_1 \leq \cdots \leq d_r$. If $r \leq 3$ or $r \geq 4$ and $d_i \geq \sigma_{i-1}$ for all $4 \leq i \leq r$, then the Hilbert series of R/I is given by*

$$P_{R/I}(z) = \left\lceil \frac{\prod_{i=1}^{r}(1 - z^{d_i})}{(1 - z)^n} \right\rceil.$$

Proof. Since Fröberg's Conjecture is known to be true if $r \leq n$, we only have to consider the case $r > n$. Let $R' = K[x_1, \cdots, x_r]$ be the polynomial ring in r variables and view R as $R = R'/(x_r, \cdots, x_{n+1})$. Then, there exist the generic homogeneous polynomials $f_1', \cdots, f_r'$ of type $(r; d_1, \cdots, d_r)$ in R' such that f_i is the image of f_i' in $R = R'/(x_r, \cdots, x_{n+1})$. Set $A = R'/(f_1', \cdots, f_r')$. It is known that A is the complete intersection and Hilbert series of A is given by

$$P_A(z) = \frac{\prod_{i=1}^{r}(1 - z^{d_i})}{(1 - z)^r}.$$

Applying Corollary 2.7.5 for $(f_1', \cdots, f_r')$ we have $x_r, \cdots, x_{n+1}, \cdots, x_1$ is a semi-regular sequence on A. By Proposition 2.7.1 we get

$$P_{A/(x_r,\cdots,x_{n+1})}(z) = \left\lceil (1 - z)^{r-n} P_A(z) \right\rceil = \left\lceil \frac{\prod_{i=1}^{r}(1 - z^{d_i})}{(1 - z)^n} \right\rceil.$$

and the theorem follows from the following isomorphisms.

$$A/(x_r, \cdots, x_{n+1}) \cong R'/(f_1', \cdots, f_r', x_r, \cdots, x_{n+1}) \cong R/(f_1, \cdots, f_r).$$

$\square$

Remark 2.7.7. Let $I = (f_1, \cdots, f_{n+1})$ be a generic ideal of type $(n; d_1, \cdots, d_{n+1})$ in $R = K[x_1, \cdots, x_n]$ with $d_1 \leq d_2 \leq \cdots \leq d_{n+1}$. If the Hilbert series of R/I is given by

$$P_{R/I}(z) = \left\lceil \frac{\prod_{i=1}^{n+1}(1 - z^{d_i})}{(1 - z)^n} \right\rceil,$$

then σ_{n+1} is the largest number such that $(R/I)_t$ is non-zero for every $t \leq \sigma_{n+1}$. Hence, in Theorem 2.7.6, the degree of f_{n+2} should be equal to σ_{n+1}. I wish to thank to Lisa Nicklasson for this remark.

3. Hilbert coefficients

In this chapter we present the basic tools of the theory of filtered modules over a Noetherian local ring. In particular we introduce the machinery we shall use throughout this thesis: $\mathbb{M}$-superficial sequences and their interplay with the Hilbert function.

3.1. **Filtered Modules and the Hilbert function.** Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d. Given an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of R, a $\mathfrak{q}$-*filtration* on M, denoted by $\mathbb{M} = \{M_n\}$, is a chain

$$\mathbb{M}: \quad M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots$$

where M_n are submodules of M and $\mathfrak{q}M_n \subseteq M_{n+1}$ for all $n \geq 0$. The $\mathfrak{q}$-filtration $\mathbb{M}$ is called a *good* $\mathfrak{q}$-*filtration* if $\mathfrak{q}M_n = M_{n+1}$ for all sufficiently large n.

Remark 3.1.1. (i) We have $\lambda(M_n/M_{n+1}) < \infty$ for all $n \geq 0$ because $\mathfrak{m}^k M_n \subseteq M_{n+1}$ for some $k > 0$.

(ii) If N is a submodule of M then $\mathbb{M}/N := \{(M_n + N)/N\}$ is a good $\mathfrak{q}$-filtration of the quotient module M/N.

Example 3.1.2. (i) The following chain

$$\mathbb{M}: \quad M \supseteq \mathfrak{q}M \supseteq \mathfrak{q}^2 M \supseteq \cdots \supseteq \mathfrak{q}^n M \supseteq \mathfrak{q}^{n+1} M \supseteq \cdots$$

is clearly a good $\mathfrak{q}$-filtration on M and it is called the $\mathfrak{q}$-adic filtration of M. In the case $M = R$ and $\mathfrak{q} = \mathfrak{m}$ we have a very important filtration of R, that is $\mathfrak{m}$-adic filtration.

(ii) Denote by $\bar{q}$ the integral closure of $\mathfrak{q}$ (see [26] for the definition and more properties of the integral closure). Then $\mathbb{F} = \{\overline{\mathfrak{q}^n}\}$ is a $\mathfrak{q}$-filtration on R and it is called the normal filtration of R with respect to $\mathfrak{q}$. A local ring $(R, \mathfrak{m})$ is said to be analytically unramified if its $\mathfrak{m}$-adic completion $\hat{R}$ is reduced. Rees in [44] showed that if R is analytically unramified then the normal filtration $\mathbb{F} = \{\overline{\mathfrak{q}^n}\}$ is a good $\mathfrak{q}$-filtration.

(iii) Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration on the module M. For every $n \geq 0$, define

$$\widetilde{M_n} := \bigcup_{k \geq 1} (M_{n+k} :_M \mathfrak{q}^k).$$

Notice that since M is Noetherian, there exists a positive integer t (depending on n) such that

$$\widetilde{M_n} = M_{n+k} : \mathfrak{q}^k \quad \forall k \geq t.$$

One can check that $\widetilde{\mathbb{M}} := \{\widetilde{M_n}\}$ is a good $\mathfrak{q}$-filtration on M and it is called the Ratliff-Rush filtration associated to $\mathbb{M}$. We refer to [42], [49] and [45, Sect. 3.1] for more properties of the Ratliff-Rush filtration. In the case $M = R$ and $\mathbb{F} = \{\widetilde{\mathfrak{q}^n}\}$, we remark that $\mathfrak{q}^n \subseteq \widetilde{\mathfrak{q}^n} \subseteq \overline{\mathfrak{q}^n}$.

Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R. The *associated graded ring* of R with respect to $\mathfrak{q}$ is a graded ring defined by

$$gr_{\mathfrak{q}}(R) = \bigoplus_{n \geq 0} \mathfrak{q}^n/\mathfrak{q}^{n+1}.$$

Recall that for every $a \in R \setminus \{0\}$ there exists n such that $a \in \mathfrak{q}^n \setminus \mathfrak{q}^{n+1}$. Then $\bar{a} \in \mathfrak{q}^n/\mathfrak{q}^{n+1}$ is called the natural image of a in $gr_{\mathfrak{q}}(R)$. For every $\bar{a} \in \mathfrak{q}^n/\mathfrak{q}^{n+1}$ and $\bar{b} \in \mathfrak{q}^m/\mathfrak{q}^{m+1}$ in $gr_{\mathfrak{q}}(R)$ we define $\bar{a}\bar{b}$ to be the image of ab in $\mathfrak{q}^{n+m}/\mathfrak{q}^{n+m+1}$.

Similarly, let M be a finitely generated R-module and $\mathbb{M} = \{M_n\}$ a good $\mathfrak{q}$-filtration on M. One can define a graded module over $gr_{\mathfrak{q}}(R)$ as follows

$$gr_{\mathbb{M}}(M) = \bigoplus_{n\geq 0} M_n/M_{n+1}$$

and it is called the *associated graded module* with respect to filtration $\mathbb{M}$. Since $\mathbb{M}$ is a good $\mathfrak{q}$-filtration on M, we have $\lambda(M_n/M_{n+1}) < \infty$ for all $n \geq 0$ (see Remark 3.1.1).

The Hilbert function of the associated graded module $gr_{\mathbb{M}}(M)$ is called the *Hilbert function* of filtration $\mathbb{M}$, by Definition 2.1.11 it is given by

$$H_{\mathbb{M}}(n) := H_{gr_{\mathbb{M}}(M)}(n) = \lambda(M_n/M_{n+1}) \text{ for all } n \geq 0.$$

The *Hilbert series* of filtration $\mathbb{M}$ is the power series

$$P_{\mathbb{M}}(t) := \sum_{n\geq 0} H_{\mathbb{M}}(n)t^n.$$

By Hilbert-Serre theorem, we can write

$$P_{\mathbb{M}}(t) = \frac{h_{\mathbb{M}}(t)}{(1-t)^d},$$

where $h_{\mathbb{M}}(t) = h_0(\mathbb{M}) + h_1(\mathbb{M})t + \ldots + h_s(\mathbb{M})t^s \in \mathbb{Z}[t]$ with $h_{\mathbb{M}}(1) \neq 0$, and $h_{\mathbb{M}}(t)$ is called the *h-polynomial* of $\mathbb{M}$. For every $i \geq 0$, we define

$$e_i(\mathbb{M}) := \frac{h_{\mathbb{M}}^{(i)}(1)}{i!}$$

and call them the *Hilbert coefficients* of filtration $\mathbb{M}$. In particular $e_0(\mathbb{M}) = h_{\mathbb{M}}(1)$ is the *multiplicity* and, by [2, Proposition 11.4] we have $e_0(\mathbb{M}) = e_0(\mathbb{N})$, for every pair of good $\mathfrak{q}$-filtrations $\mathbb{M}$ and $\mathbb{N}$ of M. One can show that for every $n \gg 0$

$$H_{\mathbb{M}}(n) = e_0(\mathbb{M})\binom{n+d-1}{d-1} - e_1(\mathbb{M})\binom{n+d-2}{d-2} + \ldots + (-1)^{d-1}e_{d-1}(\mathbb{M}).$$

Then the polynomial

$$p_{\mathbb{M}}(X) = \sum_{i=0}^{d-1}(-1)^i e_i(\mathbb{M})\binom{X+d-i-1}{d-i-1} \in \mathbb{Q}[X]$$

of degree $d-1$ is called the *Hilbert polynomial* of filtration $\mathbb{M}$ and the largest integer n such that $H_{\mathbb{M}}(n)$ and $p_{\mathbb{M}}(n)$ disagree is called the *postulation number* of filtration $\mathbb{M}$, denoted by $s(\mathbb{M})$. It turns out that $s(\mathbb{M}) = \deg(h_{\mathbb{M}}(t)) - d$ (see [7, Proposition 4.1.12]).

A rich literature has been produced on the Hilbert coefficients of a filtered module M in the case M is Cohen-Macaulay, for a survey see for instance [45]. The first Hilbert coefficient $e_1(\mathbb{M})$ is called Chern number by W.V. Vasconcelos and has been studied very deeply by several authors (see for instance [14], [25], [29], [36], [40] and [46]).

Remark 3.1.3. By definition, we can write

$$e_i(\mathbb{M}) = \sum_{k\geq i}\binom{k}{i} h_k(\mathbb{M}),$$

where $h_k(\mathbb{M}) = 0$ for every $k \geq s$. Notice that $h_k(\mathbb{M})$ are not necessarily non-negative (see Example 3.2.4).

The *Hilbert-Samuel function* of a good $\mathfrak{q}$-filtration $\mathbb{M}$ is defined by

$$H^1_{\mathbb{M}}(n) := \lambda(M/M_{n+1}) \text{ for all } n \geq 0.$$

For every $n \gg 0$ we also have

$$H^1_{\mathbb{M}}(n) = e_0(\mathbb{M})\binom{n+d}{d} - e_1(\mathbb{M})\binom{n+d-1}{d-1} + \ldots + (-1)^d e_d(\mathbb{M}).$$

In the case of the $\mathfrak{q}$-adic filtration on the ring R, we will denote by $H_{\mathfrak{q}}(n)$ the Hilbert function, by $H^1_{\mathfrak{q}}(n)$ the Hilbert-Samuel function, by $p_{\mathfrak{q}}(n)$ the Hilbert polynomial, by $s(\mathfrak{q})$ the postulation number, by $P_{\mathfrak{q}}(t)$ the Hilbert series, and by $e_i(q)$ the Hilbert coefficients of the $\mathfrak{q}$-adic filtration. In the case of the $\mathfrak{q}$-adic filtration on a module M, we will replace q with qM in the above notations.

3.2. **Superficial elements.** The notion of superficial element plays a very important role in the study of Hilbert coefficients. We only present here the definition and some basic properties of superficial elements that we use in this thesis. We refer to [45] for the existence and more interesting properties of superficial elements.

Definition 3.2.1. An element $a \in \mathfrak{q}$ is called $\mathbb{M}$-*superficial* for $\mathfrak{q}$ if there exists a non-negative integer c such that

$$(M_{n+1} :_M a) \cap M_c = M_n$$

for every $n \geq c$. A sequence $a_1, \ldots, a_r \in \mathfrak{q}$ is called $\mathbb{M}$-superficial for $\mathfrak{q}$ if for every $i = 1, \ldots, r$ the element a_i is $(\mathbb{M}/(a_1, \ldots, a_{i-1})M)$-superficial for $\mathfrak{q}$.

Suppose that the residue field $R/\mathfrak{m}$ is infinite and M is a R-module of dimension $d > 0$. Then, for every $r = 1, \ldots, d$, always there exists an $\mathbb{M}$-superficial sequence $a_1, \ldots, a_r$ for $\mathfrak{q}$. In the case $r = d$ we say $a_1, \ldots, a_d$ is a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$.

The following result is called Singh's formula because the corresponding equality in the classical case was obtained by Singh in [51].

Lemma 3.2.2. *Let $a \in \mathfrak{q}$; then for every $n \geq 0$ we have*

$$H_{\mathbb{M}}(n) = H^1_{\mathbb{M}/aM}(n) - \lambda(M_{n+1} : a/M_n).$$

Proof. The proof is an easy consequence of the following exact sequence:

$$0 \longrightarrow (M_{n+1} : a)/M_n \longrightarrow M/M_n \longrightarrow M/M_{n+1} \longrightarrow M/(M_{n+1} + aM) \longrightarrow 0.$$

□

As a corollary of Singh's formula we get a number of useful properties of superficial elements.

Proposition 3.2.3. *Let $a \in \mathfrak{q} \setminus \mathfrak{q}^2$ be an $\mathbb{M}$-superficial element for $\mathfrak{q}$ and $d = \dim(M) \geq 1$. Then we have:*

(i) $\dim(M/aM) = d - 1$.

(ii) $e_i(\mathbb{M}/aM) = e_i(\mathbb{M})$ *for* $i = 0, 1, \ldots, d-2$.

(iii) $e_{d-1}(\mathbb{M}/aM) = e_{d-1}(\mathbb{M}) + (-1)^{d-1}\lambda(0 : a)$.

(iv) *There exists an integer j such that for every $n \geq j-1$ we have*

$$e_d(\mathbb{M}/aM) = e_d(\mathbb{M}) + (-1)^d(\sum_{i=0}^{n} \lambda(M_{i+1} : a/M_i) - (n+1)\lambda(0 : a)).$$

(v) *Denote a^* the natural image of a in $gr_{\mathfrak{q}}(R)$. Then a^* is a regular element on $gr_{\mathbb{M}}(M)$ if and only if a is M-regular and $e_d(\mathbb{M}/aM) = e_d(\mathbb{M})$.*

We refer to [45, Proposition 1.2] for the proof of the above proposition.

Example 3.2.4. Let $R = K[[X,Y,Z]]/(X^3, X^2Y^3, X^2Z^4) = K[[x,y,z]]$ be a local ring of dimension 2 with the maximal ideal $\mathfrak{m}$. Consider on $M = R$ the $\mathfrak{m}$-adic filtration $\mathbb{M} = \{\mathfrak{m}^n\}$. Then the Hilbert series of $\mathbb{M}$ is

$$P_{\mathbb{M}}(t) = \frac{1+t+t^2-t^5-t^6+t^9}{(1-t)^2}$$

so that

$$e_0(\mathbb{M}) = 2, \quad e_1(\mathbb{M}) = 1, \quad e_2(\mathbb{M}) = 12.$$

We have y is an $\mathbb{M}$-superficial element for $\mathfrak{q} = \mathfrak{m}$ and

$$\lambda(\mathfrak{m}^{n+1} : y/\mathfrak{m}^n) = \begin{cases} 0 & \text{for } n = 0, \ldots, 4, \\ 1 & \text{for } n = 5, \\ 2 & \text{for } n = 6, \\ 3 & \text{for } n = 7, \\ 4 = \lambda(0 : y) & \text{for } n \geq 8 \end{cases}$$

Hence, by using Proposition 3.2.3,

$$e_0(\mathbb{M}/(y)) = 2, \quad e_1(\mathbb{M}/(y)) = -3, \quad e_2(\mathbb{M}/(y)) = -14.$$

In general it is difficult to prove that an element is superficial, the following result will be useful.

Remark 3.2.5. By Proposition 3.2.3, [45, Theorem 1.2] and Singh's formula, if $a \in \mathfrak{q}$ is an element which is M-regular, then

$$a \text{ is } \mathbb{M}\text{-superficial element for } \mathfrak{q} \Longleftrightarrow e_i(\mathbb{M}) = e_i(\mathbb{M}/aM) \text{ for every } i = 0, \ldots, d-1.$$

In the following proposition, we collect some important properties on the $\mathbb{M}$-superficial sequence for $\mathfrak{q}$ that are very useful to study the depth of M and $gr_{\mathbb{M}}(M)$.

Proposition 3.2.6. *Let $a_1, \ldots, a_r$ be an $\mathbb{M}$-superficial sequence for $\mathfrak{q}$ and $I = (a_1, \ldots, a_r)$. Then we have:*

(i) *$a_1, \ldots, a_r$ is a regular sequence on M if and only if* $\operatorname{depth} M \geq r$. (see [45, Lemma 1.2])

(ii) *$a_1^*, \ldots, a_r^*$ is a regular sequence on $gr_{\mathbb{M}}(M)$ if and only if* $\operatorname{depth} gr_{\mathbb{M}}(M) \geq r$. (see [45, Lemma 1.3])

(iii) *(Valabrega-Valla) $a_1^*, \ldots, a_r^*$ is a regular sequence on $gr_{\mathbb{M}}(M)$ if and only if $a_1, \ldots, a_r$ is a regular sequence on M and $IM \cap M_{n+1} = IM_n$ for every $n \geq 1$.* (see [45, Theorem 1.1])

(iv) *(Sally's machine)* $\operatorname{depth} gr_{\mathbb{M}/IM}(M/IM) \geq 1$ *if and only if* $\operatorname{depth} gr_{\mathbb{M}}(M) \geq r+1$. (see [45, Lemma 1.4])

3.3. **Bounds on the Hilbert coefficients in the Cohen-Macaulay case.** Let $\mathbb{M}$ be a good $\mathfrak{q}$-filtration of R-module M and J an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. If M is Cohen-Macaulay then the following inequalities hold

$$0 \leq e_0(\mathbb{M}) - \lambda(M/M_1) \leq e_1(\mathbb{M}) \leq \sum_{n \geq 0} \lambda(M_{n+1}/JM_n)$$

(see [25] and [29]) and the equalities provide good homological properties of the associated graded module $gr_{\mathbb{M}}(M) = \bigoplus_{n \geq 0} M_n/M_{n+1}$. More precisely, we have the following results.

Theorem 3.3.1. (see [29, Theorem 4.7], [45, Theorem 2.5] and [45, Theorem 2.7].) *Let $\mathbb{M} = \{M_n\}_{n \geq 0}$ be a good $\mathfrak{q}$-filtration of the Cohen–Macaulay R-module M of dimension $d \geq 1$ and let J be an ideal generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then we have:*

(i) $e_1(\mathbb{M}) \leq \sum_{n \geq 0} \lambda(M_{n+1}/JM_n)$ *with equality if and only if* $\operatorname{depth} gr_{\mathbb{M}}(M) \geq d-1$.

(ii) $e_1(\mathbb{M}) \geq \sum_{n \geq 0} \lambda(M_{n+1} + JM/JM)$ *with equality if and only if $gr_{\mathbb{M}}(M)$ is Cohen–Macaulay.*

Theorem 3.3.2. (see [45, Theorem 2.9].) *Let $\mathbb{M} = \{M_n\}_{n \geq 0}$ be a good $\mathfrak{q}$-filtration of the Cohen–Macaulay R-module M of dimension $d \geq 1$. Then the following conditions are equivalent:*

(i) $e_1(\mathbb{M}) = h_1(\mathbb{M})$.

(ii) $e_1(\mathbb{M}) = e_0(\mathbb{M}) - h_0(\mathbb{M})$.

(ii) $P_{\mathbb{M}}(t) = \frac{h_0(\mathbb{M}) + h_1(\mathbb{M})t}{(1-t)^d}$.

Moreover, if one of the above equivalent conditions holds, then $gr_{\mathbb{M}}(M)$ is Cohen-Macaulay.

If $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring, we recall that Kirby [31] proved

$$e_1(\mathfrak{m}) \leq \binom{e_0(\mathfrak{m})}{2}.$$

The result has been extended in [48] for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$. In particular if $e_0(\mathfrak{q}) \neq e_0(\mathfrak{m})$, then

$$e_1(\mathfrak{q}) \leq \binom{e_0(\mathfrak{q}) - 1}{2}.$$

By using the machinery of the theory of filtered modules, Rossi and Valla in [45] extended the above result for $\mathfrak{q}$-adic filtration on a module.

Proposition 3.3.3. (see [45, Proposition 2.10]) *Let $\mathbb{M}$ be the $\mathfrak{q}$-adic filtration on a Cohen–Macaulay module M of dimension d. Let p be an integer such that $\mathfrak{q}M \subseteq \mathfrak{m}^p M$. Then*

$$e_1(\mathbb{M}) \leq \binom{e_0(\mathbb{M}) - p + 1}{2}.$$

For completeness we recall that, by using a deeper investigation, for local Cohen–Macaulay rings of embedding dimension b and dimension d, Elias in [16, Theorem 1.6] proved

$$e_1(\mathfrak{m}) \leq \binom{e_0(\mathfrak{m})}{2} - \binom{b-d}{2}.$$

An easier approach was presented by Rossi and Valla in [47] where the result was proved for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

Theorem 3.3.4. *Let $(R,\mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal in R. Then*

$$e_1(\mathfrak{q}) \leq \binom{e_0(\mathfrak{q})}{2} - \binom{\mu(\mathfrak{q})-d}{2} - \lambda(R/\mathfrak{q}) + 1.$$

Notice that in the particular case of an $\mathfrak{m}$-primary ideal $\mathfrak{q} \subseteq \mathfrak{m}^2$ a nice proof was produced by Elias in [17].

The second Hilbert coefficient $e_2(\mathbb{M})$ has been studied by several authors as well (see for instance [9], [29], [45] and [50]). We mention here the following results.

Theorem 3.3.5. (see [45, Proposition 3.1] and [45, Theorem 2.5].) *Let $\mathbb{M} = \{M_n\}_{n\geq 0}$ be a good $\mathfrak{q}$-filtration of the Cohen–Macaulay R-module M of dimension $d \geq 1$ and let J be an ideal generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then we have:*

(i) $e_2(\mathbb{M}) \geq 0$.

(ii) $e_2(\mathbb{M}) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n)$ *with equality if and only if* $\operatorname{depth} gr_{\mathbb{M}}(M) \geq d-1$.

In the two-dimensional case, one can prove that if $e_2(M) = 0$, then $gr_{\widetilde{\mathbb{M}}}(M)$ is Cohen–Macaulay. Moreover, we have the following theorem which extend results by Sally and Narita (see [39] and [59]).

Theorem 3.3.6. (see [45, Theorem 3.1].) *Let $\mathbb{M} = \{M_n\}_{n\geq 0}$ be a good $\mathfrak{q}$-filtration of the Cohen–Macaulay R-module M of dimension 2 and let J be an ideal generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then we have:*

(i) $e_2(\mathbb{M}) \geq e_1(\mathbb{M}) - e_0(\mathbb{M}) + \lambda(M/\widetilde{M_1}) \geq 0$.

(ii) *If $e_2(\mathbb{M}) = 0$ and $M_1 = \widetilde{M_1}$, then $e_1(\mathbb{M}) = e_0(\mathbb{M}) - h_0(\mathbb{M})$ and $gr_{\mathbb{M}}(M)$ is Cohen-Macaulay.*

(iii) *$gr_{\widetilde{\mathbb{M}}}(M)$ is Cohen-Macaulay if at least one of the following conditions holds:*
  (1) $e_2(\mathbb{M}) = 0$
  (2) $e_2(\mathbb{M}) = e_1(\mathbb{M}) - e_0(\mathbb{M}) + \lambda(M/\widetilde{M_1})$ *and* $\widetilde{M_2} \cap JM = J\widetilde{M_1}$.

The above theorem was extended to dimension $d \geq 2$ by Puthenpurakal in [41]. Theorem 3.3.1 was extended to non Cohen-Macaulay case by Rossi and Valla in [46, Theorem 2.11]. In this thesis we shall extend Theorem 3.3.5 in the case module M has almost maximal depth.

3.4. **The one-dimensional case.** In this section, we would like to recall some results in the one-dimensional case which is studied in [45, Sect. 2.2]. By induction, these results are very useful in the study of the higher dimension cases.

Let M be an R-module of dimension one and $\mathbb{M} = \{M_n\}$ a good $\mathfrak{q}$-filtration. For every $n \geq 0$ define

$$u_n(\mathbb{M}) := e_0(\mathbb{M}) - H_{\mathbb{M}}(n).$$

Notice that $\dim M = 1$ implies $u_n(\mathbb{M}) = 0$ for $n \gg 0$. Moreover we have the following.

Lemma 3.4.1. [45, Lemma 2.1] *Let M be an R-module of dimension one. If a is an $\mathbb{M}$-superficial element for $\mathfrak{q}$, then for every $n \geq 0$ we have*

$$u_n(\mathbb{M}) = \lambda(M_{n+1}/aM_n) - \lambda(0 :_{M_n} a).$$

The interesting point is that we can write down the Hilbert coefficients in terms of the integers $u_n(\mathbb{M})$.

Lemma 3.4.2. [45, Lemma 2.2] *Let M be an R-module of dimension one. Then for every $i \geq 0$ we have*

$$e_i(\mathbb{M}) = \sum_{n \geq i-1} \binom{n}{i-1} u_n(\mathbb{M}).$$

Example 3.4.3. Let $R = K[[t^3, t^4, t^5]]$ be a local ring of dimension one with the maximal ideal $\mathfrak{m}$. Consider on $M = R$ the $\mathfrak{m}$-filtration $\mathbb{M} = \{M_n\}$ as the following

$$M_0 = R, \quad M_1 = \mathfrak{m}, \quad M_2 = \mathfrak{m}^2, \quad M_3 = \mathfrak{m}^2, \quad M_n = \mathfrak{m}^{n-1}$$

for $n \geq 4$. Then the Hilbert series of $\mathbb{M}$ is

$$P_{\mathbb{M}}(t) = \frac{1 + 2t - 3t^2 + 3t^3}{1-t}$$

so that

$$u_0(\mathbb{M}) = 2, \quad u_1(\mathbb{M}) = 0, \quad u_2(\mathbb{M}) = 3, \quad u_n(\mathbb{M}) = 0$$

for $n \geq 3$ and

$$e_0(\mathbb{M}) = 3, \quad e_1(\mathbb{M}) = 5, \quad e_2(\mathbb{M}) = 6.$$

Given a good $\mathfrak{q}$-filtration $\mathbb{M} = \{M_n\}_{n\geq 0}$ of the d-dimensional module M, let $a_1, \ldots, a_d$ be an $\mathbb{M}$-superficial sequence for $\mathfrak{q}$; further let $J := (a_1, ..., a_d)$ and

$$\mathbb{N} = \{J^n M\}_{n \geq 0}.$$

be the J-adic filtration on M which is clearly J-good. It is not difficult to prove that also the original filtration $\mathbb{M}$ is J-good and this implies that $e_0(\mathbb{M}) = e_0(\mathbb{N})$. It turns out that if M is Cohen-Macaulay then $a_1, \ldots, a_d$ form a regular sequence on M and $e_i(\mathbb{N}) = 0$ for every $i \geq 1$.

Denote by W the 0-th local cohomology module $H^0_{\mathfrak{m}}(M)$ of M with respect to maximal ideal $\mathfrak{m}$. We know that $H^0_{\mathfrak{m}}(M) := \cup_{j\geq 0}(0 :_M \mathfrak{m}_j) = 0 :_M m^t$ for every $t \gg 0$. In the one-dimensional case we have the following nice formula which shows that $e_1(\mathbb{N}) \leq 0$.

Lemma 3.4.4. (see [45, Lemma 2.3].) *Let M be a finitely generated R-module of dimension one and let a be a parameter for M. Then for every $t \gg 0$ we have $W = 0 :_M a^t$ and, if we denote by $\mathbb{N}$ the (a)-adic filtration on M, then*

$$e_1(\mathbb{N}) = -\lambda(W).$$

Given a good $\mathfrak{q}$-filtration $\mathbb{M} = \{M_n\}_{n\geq 0}$ of the module M (of any dimension), we consider now the corresponding filtration of the saturated module $M^{sat} := M/W$. This is the filtration

$$\mathbb{M}^{sat} := \mathbb{M}/W = \{M_n + W/W\}_{n\geq 0}.$$

Since W has finite length and $\cap_{n\geq 0} M_n = \{0\}$, we have $M_n \cap W = \{0\}$ for every $n \gg 0$. This implies $p_{\mathbb{M}}(X) = p_{\mathbb{M}^{sat}}(X)$. Furthermore we have the following result.

Proposition 3.4.5. (see [45, Proposition 2.3].) *Let M be a module of dimension d. Denote $W := H^0_{\mathfrak{m}}(M)$ and $M^{sat} := M/W$. Then*

$$e_i(\mathbb{M}) = e_i(\mathbb{M}^{sat}) \text{ for } 0 \le i \le d-1, \qquad e_d(\mathbb{M}) = e_d(\mathbb{M}^{sat}) + (-1)^d \lambda(W).$$

Notice that if $\dim(M) \ge 1$ then the module M/W always has positive depth. Therefore, in the one-dimensional case, we have that M/W is Cohen-Macaulay. This will be the strategy of the proof of the next proposition which gives the promised upper bound for e_1.

Proposition 3.4.6. (see [45, Proposition 2.4].) *Let $\mathbb{M} = \{M_n\}_{n\ge 0}$ be a good $\mathfrak{q}$-filtration of a module M of dimension one. If a is an $\mathbb{M}$-superficial element for $\mathfrak{q}$ and $\mathbb{N}$ the (a)-adic filtration on M, then*

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) \le \sum_{n\ge 0} \lambda(M_{n+1}/aM_n).$$

If $W \subseteq M_1$ and equality holds above, then M is Cohen–Macaulay.

We turn out to describing lower bounds on the first Hilbert coefficient thus extending the classical result proved by Northcott.

Proposition 3.4.7. (see [45, Proposition 2.5].) *Let $\mathbb{M} = \{M_n\}_{n\ge 0}$ be a good $\mathfrak{q}$-filtration of a module M of dimension one. If a is an $\mathbb{M}$-superficial element for $\mathfrak{q}$ and $s \ge 1$ a given integer, then for $n \gg 0$ we have*

$$\begin{aligned} e_1(\mathbb{M}) - e_1(\mathbb{N}) &= se_0(\mathbb{M}) - \lambda(M/M_s) + \lambda(M_s + W/M_s) + \lambda(M_n/a^{n-s}M_s) \\ &= \sum_{j=0}^{s-1} u_j(\mathbb{M}) + \lambda(M_s + W/M_s) + \lambda(M_n/a^{n-s}M_s). \end{aligned}$$

The following result was proved in [24, Lemma 2.1] in the case $M = R$ and $s = 1$.

Corollary 3.4.8. (see [45, Corollary 2.3].) *Let $\mathbb{M} = \{\mathfrak{q}^n M\}_{n\ge 0}$ be the $\mathfrak{q}$-adic filtration on module M of dimension one. Let $a \in \mathfrak{q}$ be an $\mathbb{M}$-superficial element for $\mathfrak{q}$ and $s \ge 1$ a given integer, then*

$$e_1(\mathbb{M}) - e_1(\mathbb{N}) = se_0(\mathbb{M}) - \lambda(M/M_s)$$

if and only if $M_{s+1} \subseteq aM_s + W$ and $W \subseteq M_s$.

3.5. **The two-dimensional case.** In this section we consider the bounds for the second Hilbert coefficient in the case M is an R-module of dimension two.

By Theorem 3.3.5 we already know that if M is Cohen-Macaulay then for every good $\mathfrak{q}$-filtration $\mathbb{M} = \{M_n\}$ of M we have

$$0 \le e_2(\mathbb{M}) \le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n),$$

where J is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. In the following theorem, we prove the upper bound for $e_2(\mathbb{M})$ as above in the case M has almost maximal depth.

Theorem 3.5.1. *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension two and* $\operatorname{depth} M > 0$. *Suppose $J = (a_1, a_2)$ is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then, we have*

$$e_2(\mathbb{M}) \le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n).$$

Further, the equality holds if and only if $\operatorname{depth} gr_{\mathbb{M}}(M) > 0$ *and* $(a_1M :_M a_2) \cap M_1 = a_1M$*.*

Proof. Define $\overline{M} := M/a_1M$ and $\{\overline{M}_n\} := \mathbb{M}/a_1M = \{(M_n + a_1M)/a_1M\}$. Then $\{\overline{M}_n\}$ is a good $\mathfrak{q}$-filtration of $\overline{M}$ and $\dim(\overline{M}) = 1$. Since $\operatorname{depth} M > 0$, by Proposition 3.2.6, one has a_1 is M-regular. Hence, from Proposition 3.2.3 (iv), we have

$$e_2(\mathbb{M}/a_1M) = e_2(\mathbb{M}) + \sum_{i=0}^{n} \lambda(M_{i+1} : a_1/M_i) \text{ for } n \gg 0. \tag{5}$$

Since a_2 is an $\mathbb{M}/a_1M$-superficial element for $\mathfrak{q}$, by Lemma 3.4.1 for every $n \geq 1$

$$\begin{aligned} u_n(\mathbb{M}/a_1M) &= \lambda(\overline{M}_{n+1}/a_2\overline{M}_n) - \lambda(a_1M :_{\overline{M}_n} a_2) \\ &= \lambda\left(\frac{M_{n+1}}{JM_n + (a_1M \cap M_{n+1})}\right) - \lambda\left(\frac{(a_1M : a_2) \cap M_n + a_1M}{a_1M}\right) \\ &\leq \lambda(M_{n+1}/JM_n). \end{aligned}$$

Therefore, by Lemma 3.4.2, we have

$$e_2(\mathbb{M}/a_1M) = \sum_{n\geq 1} n u_n(\mathbb{M}/a_1M) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n). \tag{6}$$

From (1) and (2) one has $e_2(\mathbb{M}) \leq e_2(\mathbb{M}/a_1M) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n)$.

If the equality holds then $e_2(\mathbb{M}) = e_2(\mathbb{M}/a_1M) = \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n)$. By Proposition 3.2.3 (v), $e_2(\mathbb{M}) = e_2(\mathbb{M}/a_1M)$ implies $\operatorname{depth} gr_{\mathbb{M}}(M) > 0$. Furthermore, from the inequality (2),

$$e_2(\mathbb{M}/a_1M) = \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n) \Rightarrow u_n(\mathbb{M}/a_1M) = \lambda(M_{n+1}/JM_n) \text{ for every } n \geq 1.$$

In particular, $u_1(\mathbb{M}/a_1M) = \lambda(M_2/JM_1)$ and this proves $(a_1M :_M a_2) \cap M_1 = a_1M$.

For the converse, if $\operatorname{depth} gr_{\mathbb{M}}(M) > 0$ then by Proposition 3.2.6, we have a_1^* is a regular element on $gr_{\mathbb{M}}(M)$ and $a_1M \cap M_{n+1} = a_1M_n$ for every $n \geq 1$. On the other hand, for every $n \geq 1$

$$(a_1M : a_2) \cap M_n \subseteq (a_1M : a_2) \cap M_1 = a_1M.$$

This implies $(a_1M : a_2) \cap M_n + a_1M = a_1M$. Thus for every $n \geq 1$

$$\begin{aligned} u_n(\mathbb{M}/a_1M) &= \lambda\left(\frac{M_{n+1}}{JM_n + (a_1M \cap M_{n+1})}\right) - \lambda\left(\frac{(a_1M : a_2) \cap M_n + a_1M}{a_1M}\right) \\ &= \lambda(M_{n+1}/JM_n). \end{aligned}$$

Finally, since $\operatorname{depth} gr_{\mathbb{M}}(M) > 0$, we get

$$e_2(\mathbb{M}) = e_2(\mathbb{M}/a_1M) = \sum_{n\geq 1} n u_n(\mathbb{M}/a_1M) = \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n).$$

□

Notice that if M is Cohen-Macaulay then the condition $(a_1M :_M a_2) \cap M_1 = a_1M$ holds. However, the converse is not true. For instance, we see the following examples.

Example 3.5.2. Let $R = K[x,y,z,t]/((x^2,z^2)\cap(x-y,z+t))$. Then R is a ring of dimension two and depth one. Let $J = (x^2+y^2, z^2+t^2)$ be a parameter ideal of R. Consider the J-adic filtration $\mathbb{N} = \{J^n\}$ of R. Then we have

$$((x^2+y^2) : z^2+t^2) \cap J = (x^2+y^2).$$

However, R is not Cohen-Macaulay.

Example 3.5.3. (See [34, Example 3.8]) Let $R = K[[x^5, xy^4, x^4y, y^5]] \cong K[[t_1,t_2,t_3,t_4]]/I$, where $I = (t_2t_3 - t_1t_4, t_2^4 - t_3t_4^3, t_1t_2^3 - t_3^2t_4^2, t_1^2t_2^2 - t_3^3t_4, t_1^3t_2 - t_3^4, t_3^5 - t_1^4t_4)$. Then R is a domain of dimension two and depth one. Let $J = (x^5, y^5)$ be a parameter ideal of R. Consider the J-adic filtration $\mathbb{N} = \{J^n\}$ of R. Then $e_2(\mathbb{N}) = 0$, this means the equality in Theorem 3.5.1 holds; and we are able to check that

$$((x^5) : y^5) \cap J = (x^5).$$

However, R is not Cohen-Macaulay.

Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M and assume that $\operatorname{depth} M > 0$. Let $J = (a_1, a_2)$ be an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Denote by $\mathbb{N} := \{J^nM\}$ the J-adic filtration of M. Notice that, by [45, Lemma 2.4], we can assume a_1, a_2 is also a maximal $\mathbb{N}$-superficial sequence for J. It is well known that

$$e_0(\mathbb{M}) - \lambda(M/M_1) \le e_1(\mathbb{M}) - e_1(\mathbb{N}) \le \sum_{n\ge 0} \lambda(M_{n+1}/JM_n).$$

For the second Hilbert coefficients we prove the following result.

Theorem 3.5.4. *With the above assumptions we have*

$$e_2(\mathbb{M}) - e_2(\mathbb{N}) \le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n) + \sum_{n\ge 1} \lambda(J^{n+1}M : a_1)/J^nM).$$

In particular, if $\operatorname{depth} gr_{\mathbb{N}}(M) > 0$ *then*

$$e_2(\mathbb{M}) - e_2(\mathbb{N}) \le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n).$$

Proof. Denote by $\mathbb{N}/a_1M := \{(J^nM + a_1M)/a_1M\}$ the (a_2)-adic filtration of $\overline{M} = M/a_1M$. Since a_2 is a $\mathbb{N}/a_1M$-superficial element for J, by Lemma 3.4.1 for every $n \ge 1$

$$\begin{aligned} u_n(\mathbb{N}/a_1M) &= \lambda\left(\frac{J^{n+1}M}{J^{n+1}M + (a_1M \cap J^{n+1}M)}\right) - \lambda\left(\frac{(a_1M : a_2) \cap J^nM + a_1M}{a_1M}\right) \\ &= -\lambda\left(\frac{(a_1M : a_2) \cap J^nM + a_1M}{a_1M}\right). \end{aligned}$$

Since $e_2(\mathbb{M}) \le e_2(\mathbb{M}/a_1M)$, we have

$$\begin{aligned} e_2(\mathbb{M}) - e_2(\mathbb{N}/a_1M) &\le e_2(\mathbb{M}/a_1M) - e_2(\mathbb{N}/a_1M) \\ &= \sum_{n\ge 1} n(u_n(\mathbb{M}/a_1M) - u_n(\mathbb{N}/a_1M)) \\ &\le \sum_{n\ge 1} n\lambda(M_{n+1}/JM_n). \end{aligned}$$

On the other hand, by Proposition 3.2.3 (iv) we have

$$e_2(\mathbb{N}/a_1M) = e_2(\mathbb{N}) + \sum_{n\geq 1} \lambda(J^{n+1}M : a_1)/J^nM).$$

Thus $e_2(\mathbb{M}) - e_2(\mathbb{N}) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n) + \sum_{n\geq 1} \lambda(J^{n+1}M : a_1)/J^nM)$.

In particular, if depth $gr_{\mathbb{N}}(M) > 0$ then a_1^* is $gr_{\mathbb{N}}(M)$-regular and $(J^{n+1}M : a_1) = J^nM$ for every $n \geq 1$. Hence,

$$e_2(\mathbb{M}) - e_2(\mathbb{N}) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n).$$

□

Remark 3.5.5. In the proof of Theorem 3.5.4, one observes that without assumption on the depth of $gr_{\mathbb{N}}(M)$ we have the following bound

$$e_2(\mathbb{M}) - e_2(\mathbb{N}/a_1M) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n).$$

We now consider the lower bound for the second Hilbert coefficient. In the case $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, Narita proved in [38] that $e_2(\mathfrak{q}) \geq 0$ for every $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of R. The non-negativity of the second Hilbert coefficient was extended to the case of Cohen-Macaulay modules, see for instance Theorem 3.3.5. However, in the case the module M is not Cohen-Macaulay then the second Hilbert coefficient could be negative. For instance, we see the following example.

Example 3.5.6. Let $R = K[[x, y, z]]/(x^2, xy)$ be a local ring of dimension 2 and depth $R = 1$. Then the Hilbert series of the $\mathfrak{m}$-adic filtration of R is the following

$$P_{\mathfrak{m}}(t) = \frac{1 + t - t^2}{(1 - t)^2}.$$

This means $e_2(\mathfrak{m}) = -1 < 0$.

Rossi and Valla in [45] used a very effective device to study the Hilbert coefficients that is the Ratliff-Rush filtration (see Example 3.1.2 (iii)). By using this approach we give a lower bound for the second Hilbert coefficient in the case M has almost maximal depth. More precisely, we have the following result involving the postulation number of $\widetilde{\mathbb{M}}$ (see the definition of postulation number in section 2).

Theorem 3.5.7. *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension two and* depth $M > 0$. *Suppose $J = (a_1, a_2)$ is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then, we have*

$$e_2(\mathbb{M}) \geq -\binom{s+2}{2} \lambda\left(\frac{a_1M : a_2}{a_1M}\right),$$

where s is the postulation number of the Ratliff-Rush filtration associated to $\mathbb{M}$.

Proof. Let $\widetilde{\mathbb{M}} = \{\widetilde{M_n}\}$ be the Ratliff-Rush filtration associated to $\mathbb{M}$. By [45, Lemma 3.1] we have depth $gr_{\widetilde{\mathbb{M}}}(M) \geq 1$. Hence, by Proposition 3.2.3 (v) and Lemma 3.4.2

$$e_2(\widetilde{\mathbb{M}}) = e_2(\widetilde{\mathbb{M}}/a_1M) = \sum_{n\geq 1} nu_n(\widetilde{\mathbb{M}}/a_1M).$$

By [45, Lemma 3.1], a_1, a_2 is also a maximal $\widetilde{\mathbb{M}}$-superficial sequence for $\mathfrak{q}$, so that a_2 is a $\widetilde{\mathbb{M}}/a_1M$-superficial element for $\mathfrak{q}$. Hence, by Lemma 3.4.1 for every $n \geq 1$

$$\begin{aligned} u_n(\widetilde{\mathbb{M}}/a_1M) &= \lambda(\widetilde{M}_{n+1}/J\widetilde{M}_n) - \lambda\left(\frac{(a_1M : a_2) \cap \widetilde{M}_n + a_1M}{a_1M}\right) \\ &\geq -\lambda\left(\frac{a_1M : a_2}{a_1M}\right) \end{aligned}$$

Since $\operatorname{depth} gr_{\widetilde{\mathbb{M}}}(M) \geq 1$, once has $s(\widetilde{\mathbb{M}}/a_1M) = s(\widetilde{\mathbb{M}}) + 1$. Hence $u_n(\widetilde{\mathbb{M}}/a_1M) = 0$ for every $n \geq s+2$. Thus, by [45, Lemma 3.1] we have

$$e_2(\mathbb{M}) = e_2(\widetilde{\mathbb{M}}) = \sum_{n=1}^{s+1} n u_n(\widetilde{\mathbb{M}}/a_1M) \geq -\sum_{n=1}^{s+1} n\lambda\left(\frac{a_1M : a_2}{a_1M}\right) = -\binom{s+2}{2}\lambda\left(\frac{a_1M : a_2}{a_1M}\right).$$

□

If M is Cohen-Macaulay then $(a_1M : a_2) = a_1M$. Moreover, because the study of $e_2(\mathbb{M})$ can be reduced to the 2-dimensional modules by Proposition 3.2.3, the above theorem implies the non-negativity of the second Hilbert coefficient in the Cohen-Macaulay modules as we have seen in Theorem 3.3.5.

Corollary 3.5.8. *Let $\mathbb{M}$ be a good $\mathfrak{q}$-filtration of the Cohen-Macaulay module M of dimension $d \geq 2$. Then*

$$e_2(\mathbb{M}) \geq 0.$$

3.6. **The higher dimensional case.** In this section we are going to extend Theorem 3.5.1, Theorem 3.5.4 and Theorem 3.5.7 to the higher dimensions. The following result extends Theorem 3.5.1.

Theorem 3.6.1. *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension $d \geq 2$ and $\operatorname{depth} M \geq d-1$. Suppose $J = (a_1, \ldots, a_d)$ is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. For each $i = 1, \ldots, d-1$, denote the ideal $J_i = (a_1, \ldots, a_{d-i})$ of R. Then, we have*

$$e_2(\mathbb{M}) \leq \sum_{n \geq 1} n\lambda(M_{n+1}/JM_n).$$

Further, the equality holds if and only if $\operatorname{depth} gr_{\mathbb{M}}(M) \geq d-1$ and $(J_1M :_M a_d) \cap M_1 = J_1M$.

Proof. By Theorem 3.5.1 it is enough to consider the case $d \geq 3$. Define $\overline{M} := M/J_2M$ and $\{\overline{M}_n\} := \mathbb{M}/J_2M = \{(M_n + J_2M)/J_2M\}$. Then $\{\overline{M}_n\}$ is a good $\mathfrak{q}$-filtration of $\overline{M}$ and $\dim(\overline{M}) = 2$.

Since $\operatorname{depth}(M) \geq d-1$, one has J_2 is generated by a regular sequence and $\operatorname{depth}(\overline{M}) = \operatorname{depth}(M) - (d-2) \geq 1$. Hence, by Proposition 3.2.3

$$e_2(\mathbb{M}) = e_2(\mathbb{M}/J_{d-1}M) = \ldots = e_2(\mathbb{M}/J_2M).$$

Denote by $K = (a_{d-1}, a_d)$ the ideal generated by a maximal $\mathbb{M}/J_2M$-superficial sequence for $\mathfrak{q}$, then by Theorem 3.5.1 we have

$$\begin{aligned} e_2(\mathbb{M}/J_2M) &\leq \sum_{n\geq 1} n\lambda(\overline{M}_{n+1}/K\overline{M}_n) \\ &= \sum_{n\geq 1} n\lambda\left(\frac{M_{n+1}+J_2M}{KM_n+J_2M}\right) \\ &= \sum_{n\geq 1} n\lambda\left(\frac{M_{n+1}+J_2M}{JM_n+J_2M}\right) \\ &= \sum_{n\geq 1} n\lambda\left(\frac{M_{n+1}}{JM_n+(J_2M\cap M_{n+1})}\right) \\ &\leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n) \end{aligned}$$

Thus $e_2(\mathbb{M}) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n)$.

If the equality holds then $e_2(\mathbb{M}/J_2M) = \sum_{n\geq 1} n\lambda(\overline{M}_{n+1}/K\overline{M}_n)$. By Theorem 3.5.1 we have depth $gr_{\mathbb{M}/J_2M}(M/J_2M) \geq 1$ and $(a_{d-1}\overline{M} :_{\overline{M}} a_d) \cap \overline{M}_1 = a_{d-1}\overline{M}$. Hence, by Proposition 3.2.6 (iv), we have depth $gr_{\mathbb{M}}(M) \geq d-1$. Moreover,

$$\begin{aligned} &(a_{d-1}\overline{M} :_{\overline{M}} a_d) \cap \overline{M}_1 = a_{d-1}\overline{M} \\ \Leftrightarrow\ &(J_1M/J_2M :_{\overline{M}} a_d) \cap (M_1+J_2M)/J_2M = J_1M/J_2M \\ \Leftrightarrow\ &(J_1M :_M a_d) \cap M_1 = J_1M. \end{aligned}$$

For the converse, if depth $gr_{\mathbb{M}}(M) \geq d-1$ then depth $gr_{\mathbb{M}/J_2M}(M/J_2M) \geq 1$. Hence, by Theorem 3.5.1, we have

$$e_2(\mathbb{M}/J_2M) = \sum_{n\geq 1} n\lambda(\overline{M}_{n+1}/K\overline{M}_n) = \sum_{n\geq 1} n\lambda\left(\frac{M_{n+1}}{JM_n+(J_2M\cap M_{n+1})}\right).$$

Since depth $gr_{\mathbb{M}}(M) \geq d-1$, by Proposition 3.2.6 (iii), we have

$$J_2M \cap M_{n+1} = J_2M_n \subseteq JM_n, \forall n \geq 1.$$

Thus $e_2(\mathbb{M}) = e_2(\mathbb{M}/J_2M) = \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n)$. □

As a consequence of Theorem 3.6.1 in case M is Cohen-Macaulay we get the bound for the second Hilbert coefficient given by Rossi and Valla as in Theorem 3.3.5 (ii).

Corollary 3.6.2. *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of a Cohen-Macaulay module M of dimension $d \geq 2$. Suppose J is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then, we have*

$$e_2(\mathbb{M}) \leq \sum_{n\geq 1} n\lambda(M_{n+1}/JM_n).$$

Further, the equality holds if and only if depth $gr_{\mathbb{M}}(M) \geq d-1$.

Notice that in Theorem 3.6.1 the condition depth $M \geq d-1$ is necessary as the following example shows.

Example 3.6.3. [34, Example 3.7]. Let $R = K[x, y, z, u, v, w]/I$ where I is the intersection of ideals $I = (x + y, z - u, w) \cap (z, u - v, y) \cap (x, u, w)$. Then R is a ring of dimension three and depth one. Let $\mathfrak{q} = (u - y, z + w, x - v)$ be a parameter ideal of R. Consider the $\mathfrak{q}$-adic filtration $\mathbb{N} = \{\mathfrak{q}^n\}$ of R. We have $e_2(\mathbb{N}) = 1 > 0$ and this means that the bound for e_2 in Theorem 3.6.1 is not satisfied.

Theorem 3.6.1 implies the result by Mccune on non-positivity of the second Hilbert coefficient for the parameter ideals (See [34, Theorem 3.5]).

Corollary 3.6.4. *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d - 1$. *Let $\mathfrak{q} \subseteq R$ be a parameter ideal. Then, we have $e_2(\mathfrak{q}) \leq 0$.*

Applying Theorem 3.6.1 for the $\mathfrak{m}$-adic filtration of R we have the following result.

Corollary 3.6.5. *Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d-1$. *Suppose $J = (x_1, \ldots, x_d)$ is an ideal of R generated by a maximal $\mathfrak{m}$-adic superficial sequence. Then, we have*

$$e_2(\mathfrak{m}) \leq \sum_{n \geq 1} n\lambda(\mathfrak{m}^{n+1}/J\mathfrak{m}^n).$$

Further, the equality holds if and only if R is Cohen-Macaulay and $\operatorname{depth} gr_{\mathfrak{m}}(R) \geq d - 1$.

We now extend Theorem 3.5.4 to the higher dimensions.

Theorem 3.6.6. *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension $d \geq 2$. Suppose J is an ideal of R generated by a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$ such that* $\operatorname{depth} gr_{\mathbb{N}}(M) \geq d - 1$, *where $\mathbb{N}$ is the J-adic filtration. Then, we have*

$$e_2(\mathbb{M}) - e_2(\mathbb{N}) \leq \sum_{n \geq 1} n\lambda(M_{n+1}/JM_n).$$

Proof. We proceed by induction on d. The case $d = 2$ is proved by Theorem 3.5.4. Let $d \geq 3$. By [45, Lemma 2.4] we can assume $J = (a_1, \ldots, a_d)$ where $a_1, \ldots, a_d$ is a maximal sequence of superficial elements for J with respect to $\mathbb{M}$ and $\mathbb{N}$. By Proposition 3.2.3 we have

$$e_2(\mathbb{M}/a_1M) - e_2(\mathbb{M}) = e_2(\mathbb{N}/a_1M) - e_2(\mathbb{N}) = \begin{cases} \lambda(0 : a_1) & \text{if } d = 3; \\ 0 & \text{if } d > 3. \end{cases}$$

Let $K = (a_2, \ldots, a_d)$, then K is generated by a maximal $\mathbb{M}/a_1M$-superficial sequence and $\mathbb{N}/a_1M$ is the K-adic filtration on M/a_1M. By Proposition 3.2.6, $\operatorname{depth} gr_{\mathbb{N}}(M) \geq d-1 \geq 2$ implies $\operatorname{depth} gr_{\mathbb{N}/a_1M}(M/a_1M) \geq d - 2$. Thus by induction we get

$$\begin{aligned} e_2(\mathbb{M}) - e_2(\mathbb{N}) &= e_2(\mathbb{M}/a_1M) - e_2(\mathbb{N}/a_1M) \\ &\leq \sum_{n \geq 1} n\lambda\left(\frac{M_{n+1} + a_1M}{KM_n + a_1M}\right) \\ &= \sum_{n \geq 1} n\lambda\left(\frac{M_{n+1} + a_1M}{JM_n + a_1M}\right) \\ &= \sum_{n \geq 1} n\lambda\left(\frac{M_{n+1}}{JM_n + (a_1M \cap M_{n+1})}\right) \\ &\leq \sum_{n \geq 1} n\lambda(M_{n+1}/JM_n) \end{aligned}$$

□

The following result is the generalization of Theorem 3.5.7.

Proposition 3.6.7. *Let $\mathbb{M} = \{M_n\}$ be a good $\mathfrak{q}$-filtration of R-module M of dimension $d \geq 2$ and* $\operatorname{depth} M \geq d-1$. *Suppose $a_1, a_2, \ldots, a_d$ is a maximal $\mathbb{M}$-superficial sequence for $\mathfrak{q}$. Then*

$$e_2(\mathbb{M}) \geq -\binom{s+2}{2}\lambda\left(\frac{(a_1, \ldots, a_{d-1})M : a_d}{(a_1, \ldots, a_{d-1})M}\right).$$

where s is the postulation number of the Ratliff-Rush filtration associated to $\mathbb{M}/(a_1, \ldots, a_{d-2})M$.

Proof. Define $\overline{M} = M/(a_1, \ldots, a_{d-2})M$. Then

$$\overline{\mathbb{M}} = \mathbb{M}/(a_1, \ldots, a_{d-2})M = \left\{\frac{M_n + (a_1, \ldots, a_{d-2})M}{(a_1, \ldots, a_{d-2})M}\right\}$$

is a good $\mathfrak{q}$ filtration of the 2-dimensional module $\overline{M}$. Since $K = (a_{d-1}, a_d)$ is an ideal of R generated by a maximal $\overline{\mathbb{M}}$-superficial sequence, by Theorem 3.5.7, we get

$$e_2(\mathbb{M}) = e_2(\overline{\mathbb{M}}) \geq -\binom{s+2}{2}\lambda\left(\frac{a_{d-1}\overline{M} :_{\overline{M}} a_d}{a_{d-1}\overline{M}}\right),$$

where s is a postulation number of the Ratliff-Rush filtration associated to $\overline{\mathbb{M}}$. Finally, the conclusion follows by the fact

$$\frac{(a_{d-1}\overline{M} :_{\overline{M}} a_d)}{a_{d-1}\overline{M}} \cong \frac{((a_1, \ldots, a_{d-1})M : a_d)}{(a_1, \ldots, a_{d-1})M}.$$

□

4. Deformation in local rings

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. In this chapter we study a preservation of Hilbert function of R/I under sufficient small perturbations. This study was inspired by the previous work of Srinivas and Trivedi [62] and also of Ma, Quy and Smirnov [35].

This problem was first considered by Samuel in 1956. Let $f \in R = K[[x_1, \ldots, x_d]]$ be a hypersurface with an isolated singularity, i.e. the Jacobian ideal $J(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_d})$ is $(x_1, \ldots, x_d)$-primary. Then Samuel proved that for every $\varepsilon \in (x_1, \ldots, x_d)J(f)^2$ we have an automorphism of R that maps $f \mapsto f + \varepsilon$. In particular, Samuel's result asserts that if f has an isolated singularity and ε is in a sufficiently large power of $(x_1, \ldots, x_d)$, then the rings $R/(f)$ and $R/(f + \varepsilon)$ are isomorphic. Let $\underline{f} = f_1, \ldots, f_r$ be a sequences of elements of a local ring $(R, \mathfrak{m})$, denote by $\underline{f_\varepsilon} = f_1 + \varepsilon_1, \ldots, f_r + \varepsilon_r$ a deformation (or perturbation) where $\epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$ for $N \gg 0$. As a consequence if $I = (\underline{f})$, then $I_\epsilon := (\underline{f_\varepsilon})$. Samuel's result was extended by Hironaka in 1965 to R/I an equidimensional reduced isolated singularity, provided R/I_ϵ is still reduced, equidimensional, and I_ϵ of the same height as I.

The isolated singularity is essential in the both theorems of Samuel and Hironaka. Now instead of requiring the deformation to give isomorphic rings $R/I \cong R/I_\epsilon$, we consider a weaker question: what properties and invariants are preserved by a sufficiently fine perturbation? For example, Eisenbud [15] showed how to control the homology of a complex under a perturbation and thus showed that Euler characteristic and depth can be preserved. As an application, if $\underline{f} = f_1, \ldots, f_r$ is a regular sequence, then so is the sequence $\underline{f_\varepsilon} = f_1 + \varepsilon_1, \ldots, f_r + \varepsilon_r$ as long as we take a sufficiently small perturbation. Huneke and Trivedi [27] extended this result for filter regular sequences, a generalization of the notion of regular sequences.

4.1. **Standard system of parameters.**

Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d. In this section, we define the Hilbert function of module M to be the Hilbert-Samuel function of $\mathfrak{m}$-adic filtration on M, denoted by HF_M. Thus, by definition

$$HF_M(n) := H^1_{\mathfrak{m}M}(n) = \lambda(M/\mathfrak{m}^{n+1}M)$$

for all $n \geq 0$.

Given an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of R. If M is Cohen-Macaulay and $\mathfrak{q}$ is a parameter ideal of M then $e_0(\mathfrak{q}M) = \lambda(M/\mathfrak{q}M)$. Denote $e(\mathfrak{q}M) := e_0(\mathfrak{q}M)$, in general we always have the inequality

$$e(\mathfrak{q}M) \leq \lambda(M/\mathfrak{q}M)$$

for all parameter ideals $\mathfrak{q}$ of M.

Definition 4.1.1. An R-module M is called *generalized Cohen-Macaulay* if the difference $\lambda(M/\mathfrak{q}M) - e(\mathfrak{q}M)$ is bounded above for every parameter ideal $\mathfrak{q}$ of M.

We next recall some well-known facts in the theory of generalized Cohen-Macaulay modules (see [66]).

Remark 4.1.2. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module of dimension d. For each $i \geq 0$ denote by $H^i_{\mathfrak{m}}(M)$ the i-th local cohomology module of M with support $\mathfrak{m}$. Then

(i) M is generalized Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M)$ has finite length for every $i < d$. Moreover, we have

$$\lambda(M/\mathfrak{q}M) - e(\mathfrak{q}M) \leq \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H^i_{\mathfrak{m}}(M))$$

for all parameter ideals $\mathfrak{q}$. The left hand side, denoted by $I(M)$, is called the *Buchsbaum invariant* of M.

(ii) If M is generalized Cohen-Macaulay, then for every part of system of parameters $x_1, \ldots, x_r$ we have $I(M) \geq I(M/(x_1, \ldots, x_r)M)$.

Definition 4.1.3. A sequence $x_1, \ldots, x_t$ in $\mathfrak{m}$ is called a *filter regular sequence* of M if

$$\mathrm{Supp}\left(\frac{(x_1, \ldots, x_{i-1})M : x_i}{(x_1, \ldots, x_{i-1})M}\right) \subseteq \{\mathfrak{m}\}$$

for all $i = 1, \ldots, t$.

We notice that if M is generalized Cohen-Macaulay, then every system of parameter is a filter regular sequence of M.

Definition 4.1.4. Let M be a generalized Cohen-Macalay module of dimension d. A parameter ideal $\mathfrak{q}$ of M is called *standard* if

$$\lambda(M/\mathfrak{q}M) - e(\mathfrak{q}M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H^i_{\mathfrak{m}}(M)).$$

Remark 4.1.5. Let M be a generalized Cohen-Macalay module of dimension d. Then there exists a positive integer N such that $\mathfrak{q}$ is standard for every parameter ideal $\mathfrak{q} \subseteq \mathfrak{m}^N$. In fact we can choose $N = I(M)$.

The Hilbert-Samuel function of a standard parameter ideal $\mathfrak{q}$ can be expressed explicitly as follows (see [66, Corollary 4.2]).

Theorem 4.1.6. *$\mathfrak{q}$ is a standard parameter ideal of M if and only if*

$$HF_{\mathfrak{q}M}(n) = \binom{n+d}{d} e(\mathfrak{q}M) + \sum_{i=1}^{d} \sum_{j=0}^{d-i} \binom{n+d-i}{d-i} \binom{d-i-1}{j-1} \lambda(H^j_{\mathfrak{m}}(M)),$$

for all $n \geq 0$.

4.2. **Extended degrees.** In order to capture the complexity of non (generalized) Cohen-Macaulay modules, Vasconcelos et al. [68, 69] introduced the notion of extended degree which is a generalization of the notion of multiplicity. Let $\mathcal{M}(R)$ be the category of finitely generated R-modules. An *extended degree* on $\mathcal{M}(R)$ is a numerical function $D(\bullet)$ on $\mathcal{M}(R)$ such that the following properties hold for every R-module $M \in \mathcal{M}(R)$:

(i) $D(M) = D(M/L) + \lambda(L)$, where $L = H^0_{\mathfrak{m}}(M)$,

(ii) $D(M) \geq D(M/xM)$ for a generic element x of $\mathfrak{m}$,

(iii) $D(M) = e(\mathfrak{m}M)$ if M is a Cohen-Macaulay module, where $e(\mathfrak{m}M) = e_0(\mathfrak{m}M)$ is the multiplicity of module M.

The prototype of an extended degree is the homological degree defined by Vasconcelos in [68]. If R is a homomorphic image of a Gorenstein ring S with $\dim S = n$ then the *homological* *degree* of R-module M is defined by

$$\operatorname{hdeg}(M) := e(\mathfrak{m}M) + \sum_{i=0}^{d-1} \binom{d-1}{i} \operatorname{hdeg}(\operatorname{Ext}_S^{n-i}(M,S)).$$

Recently, Cuong and Quy [13] introduced a new extended degree called the *unmixed degree*, and denoted by $\operatorname{udeg}(M)$. The readers are encouraged to [13] for more details about the construction. If M is generalized Cohen-Macaulay then

$$\operatorname{hdeg}(M) = \operatorname{udeg}(M) = e(\mathfrak{m}M) + I(M).$$

We now present some lemmas that will be useful for the proof of the main results in the section 4.

Lemma 4.2.1. *Let $(R,\mathfrak{m})$ be a local ring of dimension d and $\underline{f} = f_1,\ldots,f_r$ a part of system of parameters. Let $D(\bullet)$ be an extended degree and set $N = D(R/(\underline{f})) + 1$. Then for every $\underline{\varepsilon} = \varepsilon_1,\ldots,\varepsilon_r \in \mathfrak{m}^N$ the sequence $\underline{f_\varepsilon} = f_1+\varepsilon_1,\ldots,f_r+\varepsilon_r$ is a part of system of parameters of R.*

Proof. Let $x_1,\ldots,x_{d-r}$ be a general sequence of elements of $R/(\underline{f})$. Then

$$\lambda(R/(\underline{f},x_1,\ldots,x_{d-r})) \le D(R/(\underline{f})) = N-1.$$

This implies that $\mathfrak{m}^{N-1} \subseteq (\underline{f},x_1,\ldots,x_{d-r})$. Therefore for every $\underline{\varepsilon} = \varepsilon_1,\ldots,\varepsilon_r \in \mathfrak{m}^N$ we have

$$(\underline{f},x_1,\ldots,x_{d-r}) = (\underline{f_\varepsilon},x_1,\ldots,x_{d-r}).$$

Hence $\underline{f_\varepsilon},x_1,\ldots,x_{d-r}$ is a system of parameters of R. The proof is complete. ☐

Lemma 4.2.2. [68, Corollary 3.6] *Let $(R,\mathfrak{m})$ be a local ring of dimension d with infinite residue field. Let $\operatorname{hdeg}(\bullet)$ be the homological degree. Then there exists a minimal reduction J of $\mathfrak{m}$ with reduction number $\mathrm{r}_J(\mathfrak{m}) \le d!\,\operatorname{hdeg}(R)-1$.*

Lemma 4.2.3. *Let $(R,\mathfrak{m})$ be a local ring of dimension d and $\underline{f} = f_1,\ldots,f_r$ a part of system of parameters. Let J be a minimal reduction of $\mathfrak{m}$ in $R/(\underline{f})$, and k a non-negative integer such that $\mathrm{r}_J(\mathfrak{m},R/(\underline{f})) \le k$. Then for every $\underline{\varepsilon} = \varepsilon_1,\ldots,\varepsilon_r \in \mathfrak{m}^{k+2}$, J is a minimal reduction of $\mathfrak{m}$ in $R/(\underline{f_\varepsilon})$ and $\mathrm{r}_J(\mathfrak{m},R/(\underline{f_\varepsilon})) \le k+1$.*

Proof. By the assumption we have $\mathfrak{m}^{k+1} + (\underline{f}) = J\mathfrak{m}^k + (\underline{f})$. Therefore $\mathfrak{m}^{k+1} \subseteq J + (\underline{f})$. Hence for every $\underline{\varepsilon} = \varepsilon_1,\ldots,\varepsilon_r \in \mathfrak{m}^{k+2}$ we have $J + (\underline{f}) = J + (\underline{f_\varepsilon})$. We are going to prove that $\mathfrak{m}^{k+2} + (\underline{f_\varepsilon}) = J\mathfrak{m}^{k+1} + (\underline{f_\varepsilon})$. We have

$$\begin{aligned}
J\mathfrak{m}^{k+1} + (\underline{f_\varepsilon}) + \mathfrak{m}(\mathfrak{m}^{k+2} + (\underline{f_\varepsilon})) &= J\mathfrak{m}^{k+1} + (\underline{f_\varepsilon}) + \mathfrak{m}(\mathfrak{m}^{k+2} + (\underline{f}))\\
&= J\mathfrak{m}^{k+1} + (\underline{f_\varepsilon}) + \mathfrak{m}(J\mathfrak{m}^{k+1} + (\underline{f}))\\
&= J\mathfrak{m}^{k+1} + (\underline{f_\varepsilon}) + \mathfrak{m}(\underline{f})\\
&= \mathfrak{m}(J\mathfrak{m}^{k} + (\underline{f})) + (\underline{f_\varepsilon})\\
&= \mathfrak{m}(\mathfrak{m}^{k+1} + (\underline{f})) + (\underline{f_\varepsilon})\\
&= \mathfrak{m}^{k+2} + (\underline{f_\varepsilon}) + \mathfrak{m}(\underline{f})\\
&= \mathfrak{m}^{k+2} + (\underline{f_\varepsilon}).
\end{aligned}$$

The last equality follows from the fact that $\mathfrak{m}(\underline{f}) \subseteq \mathfrak{m}^{k+2} + (\underline{f}) = \mathfrak{m}^{k+2} + (\underline{f_\varepsilon})$. By NAK we have $\mathfrak{m}^{k+2} + (\underline{f_\varepsilon}) = J\mathfrak{m}^{k+1} + (\underline{f_\varepsilon})$. The proof is complete. ☐

4.3. **Deformation in local rings.** Srinivas and Trivedi [62] showed that the Hilbert function of a sufficiently fine perturbation is bounded above by the original Hilbert function. Furthermore they proved that the Hilbert functions of $R/(f_1, \ldots, f_r)$ and $R/(f_1 + \varepsilon_1, \ldots, f_r + \varepsilon_r)$ coincide under small perturbations provided two conditions: (a) $\underline{f} = f_1, \ldots, f_r$ a filter regular sequence; (b) $R/(f_1, \ldots, f_r)$ is generalized Cohen-Macaulay. Srinivas and Trivedi gave examples to show that we can not remove the condition (a) even if $\underline{f} = f_1, \ldots, f_r$ is a part of system of parameters. However they asked whether the condition (b) is superfluous. Recently, Ma, Quy and Smirnov [35] answered this question and proved the following.

Theorem 4.3.1. (see [35, Theorem 14].) *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d, and $\underline{f} = f_1, \ldots, f_r$ a filter regular sequence. Then there exists $N > 0$ such that for every $\underline{\varepsilon} = \varepsilon_1, \ldots, \varepsilon_r \in \mathfrak{m}^N$, the Hilbert-Samuel functions of $R/(f_1, \ldots, f_r)$ and $R/(f_1 + \varepsilon_1, \ldots, f_r + \varepsilon_r)$ are equal, i.e.*

$$\lambda\left(\frac{R}{\mathfrak{m}^n + (\underline{f})}\right) = \lambda\left(\frac{R}{\mathfrak{m}^n + (\underline{f_\epsilon})}\right)$$

for all $n \geq 1$.

We also asked the question.

Question 4.3.2. Can one obtain explicit bounds on N?

A positive answer for the case $r = 1$ was given in [35, Theorem 3.3]. For any r and R is a Cohen-Macaulay local ring of dimension d, Srinivas and Trivedi [63, Proposition 1.1] provided a formula for N in terms of the multiplicity. Namely, we can choose

$$N = (d - r)!\, e(\mathfrak{m}R/(f_1, \ldots, f_r)) + 2.$$

Inspired by the above formula, one can hope to give a bound for N in any local ring by using the extended degree instead of the multiplicity. We will extend the above result of Srinivas and Trivedi for the class of generalized Cohen-Macaulay rings by using the multiplicity and the length of local cohomology $H^i_{\mathfrak{m}}(R)$.

Let $(R, \mathfrak{m})$ be a local ring and M a generalized Cohen-Macaulay module of dimension d. The Buchsbaum invariant of M is defined as follows

$$I(M) = \sum_{i=0}^{d-1} \binom{d-1}{i} \lambda(H^i_{\mathfrak{m}}(M)).$$

We proved the following result.

Theorem 4.3.3. (see Theorem 4.5.2.) *Let $(R, \mathfrak{m})$ be a generlized Cohen-Macaulay local ring of dimension d, and $\underline{f} = f_1, \ldots, f_r$ a part of system of parameters of R. Let $s = d - r$, and*

$$N = s!\left(e(\mathfrak{m}R/(\underline{f})) + I(R/(\underline{f}))\right) + (s+1)I(R) + 1.$$

Then for every $\underline{\epsilon} = \epsilon_1, \cdots, \epsilon_r \in \mathfrak{m}^N$ we have the Hilbert-Samuel functions of $R/(\underline{f})$ and $R/(\underline{f_\varepsilon})$ are equal.

The method of our proof of the above result is similar to the Srinivas and Trivedi one in the Cohen-Macaulay case. Let us mention the most important step in our proof. If R is Cohen-Macaulay and $J = (x_1, \ldots, x_s)$ a minimal reduction of $\mathfrak{m}$ with respect to $R/(\underline{f})$, then we can choose N such that $J + (\underline{f}) = J + (\underline{f_\varepsilon})$ for any $\underline{\varepsilon} = \varepsilon_1, \ldots, \varepsilon_r \in \mathfrak{m}^N$. The strategy of Srinivas and Trivedi is to transfer the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_\varepsilon})$ with respect to

$\mathfrak{m}$ to the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_\varepsilon})$ with respect to the parameter ideal J, and using the following well-known fact for Cohen-Macaulay rings

$$\lambda(R/(J^{n+1},\underline{f})) = \binom{n+s}{s}\lambda(R/(J,\underline{f})) = \binom{n+s}{s}\lambda(R/(J,\underline{f_\varepsilon})) = \lambda(R/(J^{n+1},\underline{f_\varepsilon})).$$

For generalized Cohen-Macaulay rings, we also have an explicit formula for the Hilbert function with respect to special parameter ideals, say standard parameter ideals, in terms of the length of lower local cohomology modules (see Theorem 4.1.6). Therefore we need to control $\lambda(H^i_{\mathfrak{m}}(R/(\underline{f}))$ under sufficiently small perturbations. This fact motivates us to prove the following result.

Theorem 4.3.4. (see Theorem 4.4.2.) *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d and $\underline{f} = f_1,\cdots,f_r$ a part of system of parameters. Let $N = e(\mathfrak{m}R/(\underline{f}))+I(R)+1$, then for every $\underline{\epsilon} = \epsilon_1,\cdots,\epsilon_r \in \mathfrak{m}^N$ we have*

$$\lambda(H^i_{\mathfrak{m}}(R/(\underline{f}))) = \lambda(H^i_{\mathfrak{m}}(R/(\underline{f_\varepsilon})))$$

for every $0 \le i < d-r$.

4.4. **The local cohomology under small perturbations.** Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d and $\underline{f} = f_1,\ldots,f_r$ a part of system of parameters. In this section we provide a positive integer N depends on $e(\mathfrak{m}R/(\underline{f}))$ and $I(R)$ such that for every $\underline{\varepsilon} = \epsilon_1,\ldots,\epsilon_r \in \mathfrak{m}^N$ the lengths of $H^i_{\mathfrak{m}}(R/(\underline{f}))$ and $H^i_{\mathfrak{m}}(R/(\underline{f_\varepsilon}))$ coincide for every $0 \le i < d-r$.

The proof of the main result is based on the induction on r, where r is the length of the sequence $\underline{f}$. First, for the case $r=1$ we have the following proposition.

Proposition 4.4.1. *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d, and f a parameter element of R. Then for every $\epsilon \in \mathfrak{m}^{I(R)}$ such that $f+\epsilon$ is a parameter element of R, we have*

$$\lambda(H^i_{\mathfrak{m}}(R/(f))) = \lambda(H^i_{\mathfrak{m}}(R/(f+\epsilon)))$$

for every $i < d-1$.

Proof. Following from the short exact sequence

$$0 \to R/(0:f) \xrightarrow{f} R \to R/(f) \to 0$$

we obtain the following short exact sequence

$$0 \to \frac{H^i_{\mathfrak{m}}(R)}{fH^i_{\mathfrak{m}}(R)} \to H^i_{\mathfrak{m}}(R/(f)) \to (0:_{H^{i+1}_{\mathfrak{m}}(R)} f) \to 0$$

for every $i < d-1$. Similarly we have the following short exact sequence

$$0 \to \frac{H^i_{\mathfrak{m}}(R)}{(f+\epsilon)H^i_{\mathfrak{m}}(R)} \to H^i_{\mathfrak{m}}(R/(f+\epsilon)) \to (0:_{H^{i+1}_{\mathfrak{m}}(R)} (f+\epsilon)) \to 0$$

for every $i < d-1$. Since $\epsilon \in \mathfrak{m}^{I(R)}$ we have $\epsilon H^i_{\mathfrak{m}}(R) = 0$ for all $i < d$. It follows that $(0:_{H^{i+1}_{\mathfrak{m}}(R)} f) \cong (0:_{H^{i+1}_{\mathfrak{m}}(R)} (f+\epsilon))$ and $\frac{H^i_{\mathfrak{m}}(R)}{fH^i_{\mathfrak{m}}(R)} \cong \frac{H^i_{\mathfrak{m}}(R)}{(f+\epsilon)H^i_{\mathfrak{m}}(R)}$ for all $i < d-1$. Hence the above two short exact sequences imply

$$\lambda(H^i_{\mathfrak{m}}(R/(f))) = \lambda(H^i_{\mathfrak{m}}(R/(f+\epsilon)))$$

for all $i < d-1$. $\square$

We now present the main result of this section.

Theorem 4.4.2. *Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d and $\underline{f} = f_1, \ldots, f_r$ a part of system of parameters. Let $N = e(\mathfrak{m}R/(\underline{f})) + I(R) + 1$, then for every $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$ and $i < d - r$ we have*

$$\lambda(H^i_{\mathfrak{m}}(R/(\underline{f}))) = \lambda(H^i_{\mathfrak{m}}(R/(\underline{f_\varepsilon}))).$$

Proof. Without loss of generality we will always assume that the residue field is infinite. We proceed by induction on r. For $r = 1$, we have

$$N \geq e(\mathfrak{m}R/(\underline{f})) + I(R/(\underline{f})) + 1 = \mathrm{hdeg}(R/(\underline{f})) + 1.$$

So $f_1 + \epsilon_1$ is a parameter element by Lemma 4.2.1. Hence we are done by Proposition 4.4.1.

For $r > 1$, let $R_1 = R/(f_1)$. For simplicity, we will identify f_i with its image in R_1. Since $N \geq e(\mathfrak{m}R_1/(f_2, \ldots, f_r)R_1) + I(R_1) + 1$ and $\epsilon_2, \ldots, \epsilon_r \in \mathfrak{m}^N$, by induction we get

$$\lambda(H^i_{\mathfrak{m}}(R/(f_1, f_2, \ldots, f_r))) = \lambda(H^i_{\mathfrak{m}}(R/(f_1, f_2 + \epsilon_2, \ldots, f_r + \epsilon_r)))$$

for every $i < d-r$. Since $N \geq \mathrm{hdeg}(R/(\underline{f}))+1$ and $\epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$, by Lemma 4.2.1 we have $f_1, f_2 + \epsilon_2, \ldots, f_r + \epsilon_r$ and $f_1 + \epsilon_1, f_2 + \epsilon_2, \ldots, f_r + \epsilon_r$ are the parts of system of parameters of R. Let $R_2 = R/(f_2 + \epsilon_2, \ldots, f_r + \epsilon_r)$, we have f_1 and $f_1 + \epsilon_1$ are parameter elements of R_2. Moreover, $N \geq I(R_2)$ and $\epsilon_1 \in \mathfrak{m}^N$, by Proposition 4.4.1 we get

$$\lambda(H^i_{\mathfrak{m}}(R_2/f_1R_2)) = \lambda(H^i_{\mathfrak{m}}(R_2/(f_1 + \epsilon_1)R_2))$$

for every $i < d - r$. That is

$$\lambda(H^i_{\mathfrak{m}}(R/(f_1, f_2 + \epsilon_2, \ldots, f_r + \epsilon_r))) = \lambda(H^i_{\mathfrak{m}}(R/(f_1 + \epsilon_1, f_2 + \epsilon_2, \ldots, f_r + \epsilon_r)))$$

for every $i < d - r$. Hence we obtain the desired assertion. The proof is complete. $\square$

4.5. **The Hilbert function under small perturbations.** In this section, let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d, and $\underline{f} = f_1, \ldots, f_r$ a part of system of parameters. We will find an explicitly positive integer N depends on $\mathrm{hdeg}(R/(\underline{f}))$ and $I(R)$ such that for every $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$ the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_\epsilon})$ coincide. The following lemma is a special case of Lemma 4.2.2 and Lemma 4.2.3.

Lemma 4.5.1. *Let $(R, \mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d with the infinite residue field, and $\underline{f} = f_1, \ldots, f_r$ a part of system of parameters. Let $s = d - r$, and $C = s!\,\mathrm{hdeg}(R/(\underline{f})) + 1$. Then there exists a minimal reduction J of $\mathfrak{m}$ in $R/(\underline{f})$ such that*

1. $\mathfrak{m}^{C+k} + (\underline{f}) = J^{k+1}\mathfrak{m}^{C-1} + (\underline{f})$ *for all $k \geq 0$.*
2. *For every $\underline{\varepsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^C$ one has $\mathfrak{m}^{C+k} + (\underline{f_\varepsilon}) = J^{k+1}\mathfrak{m}^{C-1} + (\underline{f_\varepsilon})$ for all $k \geq 0$.*

Proof. By Lemma 4.2.2 there exists a minimal reduction $J = (x_1, \ldots, x_{d-r})$ of $\mathfrak{m}$ in $R/(\underline{f})$ such that $\mathrm{r}_J(\mathfrak{m}, R/(\underline{f})) \leq (d-r)!\,\mathrm{hdeg}(R/(\underline{f})) - 1 = C - 2$. Hence $\mathfrak{m}^C + (\underline{f}) = J\mathfrak{m}^{C-1} + (\underline{f})$. By Lemma 4.2.3 one has J is a minimal reduction of $\mathfrak{m}$ in $R/(\underline{f_\epsilon})$ and $\mathrm{r}_J(\mathfrak{m}, R/(\underline{f_\epsilon})) \leq C - 1$. Hence $\mathfrak{m}^C + (\underline{f_\epsilon}) = J\mathfrak{m}^{C-1} + (\underline{f_\epsilon})$ for every $\underline{\varepsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^C$. Therefore, for all $k \geq 0$ we have $\mathfrak{m}^{C+k}R/(\underline{f}) = J^{k+1}\mathfrak{m}^{C-1}R/(\underline{f})$ and $\mathfrak{m}^{C+k}R/(\underline{f_\varepsilon}) = J^{k+1}\mathfrak{m}^{C-1}R/(\underline{f_\varepsilon})$. The claims are now clear. $\square$

The following theorem is the main result of this chapter. It extends the result of Srinivas and Trivedi [63, Proposition 1] for generalized Cohen-Macaulay rings.

Theorem 4.5.2. *Let $(R,\mathfrak{m})$ be a generalized Cohen-Macaulay ring of dimension d, and $\underline{f} = f_1, \ldots, f_r$ a part of system of parameters. Let $s = d - r$, and*

$$N = s!\ \mathrm{hdeg}(R/(\underline{f})) + (s+1)I(R) + 1.$$

Then for every $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$ the Hilbert functions of $R/(\underline{f})$ and $R/(\underline{f_\epsilon})$ are equal, i.e.

$$\lambda\left(\frac{R}{\mathfrak{m}^n + (\underline{f})}\right) = \lambda\left(\frac{R}{\mathfrak{m}^n + (\underline{f_\epsilon})}\right)$$

for all $n \geq 1$.

Proof. Without loss of generality we may assume that the residue field is infinite. Let $C = s!\ \mathrm{hdeg}(R/(\underline{f})) + 1$, by Lemma 4.5.1 there exists ideal $J = (x_1, \ldots, x_s) \subseteq \mathfrak{m}$ such that

$$\mathfrak{m}^{C+k} + (\underline{f}) = J^{k+1}\mathfrak{m}^{C-1} + (\underline{f})$$

for all $k \geq 0$. Moreover, since $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N \subseteq \mathfrak{m}^C$ we also have

$$\mathfrak{m}^{C+k} + (\underline{f_\epsilon}) = J^{k+1}\mathfrak{m}^{C-1} + (\underline{f_\epsilon})$$

for all $k \geq 0$. Let $t = \max\{I(R), 1\}$. For all $0 \leq i \leq t-1$ set $N_i = C + s(t-1) + i$. We have $N_i \leq N$ for all $i \leq t-1$. Since $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N \subseteq \mathfrak{m}^{N_0}$ we have

$$\mathfrak{m}^i + (\underline{f}) = \mathfrak{m}^i + (\underline{f_\epsilon})$$

for all $i \leq N_0$. Therefore, it is enough to prove that

$$\lambda\left(\frac{R}{\mathfrak{m}^n + (\underline{f})}\right) = \lambda\left(\frac{R}{\mathfrak{m}^n + (\underline{f_\epsilon})}\right) \qquad (1)$$

for all $n \geq N_0$. We will prove it in the following equivalent form

$$\lambda\left(\frac{R}{\mathfrak{m}^{N_i+kt} + (\underline{f})}\right) = \lambda\left(\frac{R}{\mathfrak{m}^{N_i+kt} + (\underline{f_\epsilon})}\right) \qquad (2)$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$. Set $\tilde{J} = (x_1^t, \ldots, x_s^t)$, one has $\tilde{J} \subseteq \mathfrak{m}^t$ is a standard parameter ideal of $R/(\underline{f})$ and $\tilde{J}J^{(s-1)(t-1)} = J^{s(t-1)+1}$.

Claim 1. For all $0 \leq i \leq t-1$ and all $k \geq 0$ we have

$$\mathfrak{m}^{N_i+kt} + (\underline{f}) = \tilde{J}^{k+1}\mathfrak{m}^{N_i-t} + (\underline{f}),$$

and

$$\mathfrak{m}^{N_i+kt} + (\underline{f_\varepsilon}) = \tilde{J}^{k+1}\mathfrak{m}^{N_i-t} + (\underline{f_\varepsilon})$$

for all $\underline{\epsilon} = \epsilon_1, \ldots, \epsilon_r \in \mathfrak{m}^N$.

Proof of Claim 1. We have

$$\begin{aligned}
\mathfrak{m}^{N_i+kt}R/(\underline{f}) &= J^{s(t-1)+kt+i+1}\mathfrak{m}^{C-1}R/(\underline{f}) \\
&= \tilde{J}^{k+1}J^{(s-1)(t-1)+i}\mathfrak{m}^{C-1}R/(\underline{f}) \\
&= \tilde{J}^{k+1}\mathfrak{m}^{N_i-t}R/(\underline{f})
\end{aligned}$$

for all $k \geq 0$. Therefore

$$\mathfrak{m}^{N_i+kt} + (\underline{f}) = \tilde{J}^{k+1}\mathfrak{m}^{N_i-t} + (\underline{f})$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$. The second assertion can be proved similarly. The Claim is proved. $\square$

By Claim 1, in order to prove the equality (2) it is enough to show

$$\lambda\left(\frac{R}{\tilde{J}^{k+1}\mathfrak{m}^{N_i-t}+(\underline{f})}\right)=\lambda\left(\frac{R}{\tilde{J}^{k+1}\mathfrak{m}^{N_i-t}+(\underline{f_\varepsilon})}\right) \qquad (3)$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$. On the other hand, since $N \geq e(\mathfrak{m}R/(\underline{f})) + I(R) + 1$ we have

$$\lambda(H^i_{\mathfrak{m}}(R/(\underline{f}))) = \lambda(H^i_{\mathfrak{m}}(R/(\underline{f_\epsilon})))$$

for all $i < s$ by Theorem 4.4.2. We also have $(\underline{f}) + \tilde{J} = (\underline{f_\epsilon}) + \tilde{J}$ since $(\underline{f}) + \tilde{J} \supseteq \mathfrak{m}^{N-1}$. Hence $\tilde{J}$ is a standard parameter ideal of $R/(\underline{f_\epsilon})$ and $e(\widetilde{J}R/(\underline{f_\epsilon})) = e(\widetilde{J}R/(\underline{f}))$. By Theorem 4.1.6 we have

$$\lambda\left(\frac{R}{\tilde{J}^{k+1}+(\underline{f})}\right)=\lambda\left(\frac{R}{\tilde{J}^{k+1}+(\underline{f_\varepsilon})}\right) \qquad (4)$$

for all $k \geq 0$. Therefore in order to prove the equality (3) it is sufficient to prove that

$$\lambda\left(\frac{\tilde{J}^{k+1}+(\underline{f})}{\tilde{J}^{k+1}\mathfrak{m}^{N_i-t}+(\underline{f})}\right)=\lambda\left(\frac{\tilde{J}^{k+1}+(\underline{f_\varepsilon})}{\tilde{J}^{k+1}\mathfrak{m}^{N_i-t}+(\underline{f_\varepsilon})}\right) \qquad (5)$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$. Let $L = (\underline{f}) + \tilde{J} = (\underline{f_\epsilon}) + \tilde{J}$. We will prove (5) in the following equivalent form

$$\lambda\left(\frac{L^{k+1}+(\underline{f})}{L^{k+1}\mathfrak{m}^{N_i-t}+(\underline{f})}\right)=\lambda\left(\frac{L^{k+1}+(\underline{f_\varepsilon})}{L^{k+1}\mathfrak{m}^{N_i-t}+(\underline{f_\varepsilon})}\right) \qquad (6)$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$.

Claim 2. For all $k \geq 0$ we have $L^{k+1} \cap (\underline{f}) = (\underline{f})L^k$ and $L^{k+1} \cap (\underline{f_\epsilon}) = (\underline{f_\epsilon})L^k$.

Proof of Claim 2. Notice that $x_1^t, \ldots, x_s^t$ forms a d-sequence of $R/(\underline{f})$, by [25, Theorem 2.1] we have

$$\tilde{J}^{k+1} \cap (\underline{f}) \subseteq \tilde{J}^k(\underline{f}).$$

Hence, for all $k \geq 0$ we have

$$\begin{aligned} L^{k+1} \cap (\underline{f}) &= ((\underline{f}) + \tilde{J})^{k+1} \cap (\underline{f}) \\ &= ((\underline{f})L^k + \tilde{J}^{k+1}) \cap (\underline{f}) \\ &= (\underline{f})L^k + \tilde{J}^{k+1} \cap (\underline{f}) \\ &= (\underline{f})L^k. \end{aligned}$$

The second assertion can be proved similarly. $\square$

We continue the proof of our theorem. Follows from Claim 2 we have

$$\begin{aligned} L^{k+1}\mathfrak{m}^{N_i-t} + L^{k+1} \cap (\underline{f}) &= L^{k+1}\mathfrak{m}^{N_i-t} + L^k(\underline{f}) \\ &= L^k(L\mathfrak{m}^{N_i-t} + (\underline{f})) \\ &= L^k(\tilde{J}\mathfrak{m}^{N_i-t} + (\underline{f})) \\ &= L^k(\mathfrak{m}^{N_i} + (\underline{f})) \\ &= L^k(\mathfrak{m}^{N_i} + (\underline{f_\epsilon})) \\ &= L^k(\tilde{J}\mathfrak{m}^{N_i-t} + (\underline{f_\epsilon})) \\ &= L^k(L\mathfrak{m}^{N_i-t} + (\underline{f_\epsilon})) \\ &= L^{k+1}\mathfrak{m}^{N_i-t} + L^{k+1} \cap (\underline{f_\epsilon}) \end{aligned}$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$. Therefore

$$\begin{aligned} \frac{L^{k+1} + (\underline{f})}{L^{k+1}\mathfrak{m}^{N_i-t} + (\underline{f})} &\cong \frac{L^{k+1}}{L^{k+1}\mathfrak{m}^{N_i-t} + L^{k+1} \cap (\underline{f})} \\ &\cong \frac{L^{k+1}}{L^{k+1}\mathfrak{m}^{N_i-t} + L^{k+1} \cap (\underline{f_\epsilon})} \\ &\cong \frac{L^{k+1} + (\underline{f_\varepsilon})}{L^{k+1}\mathfrak{m}^{N_i-t} + (\underline{f_\varepsilon})} \end{aligned}$$

for all $0 \leq i \leq t-1$ and all $k \geq 0$. The equality (6) is now clear. The proof is complete. $\square$

We close this section with the following remark.

Remark 4.5.3. If R is Cohen-Macaulay our formula $N = s!\, e(\mathfrak{m}R/(\underline{f})) + 1$ is slightly better than the above mentioned. If R is generalized Cohen-Macaulay but not Cohen-Macaulay, according to the proof we can choose

$$N = N_{t-1} = s!\ \mathrm{hdeg}(R/(\underline{f})) + (s+1)I(R) - s.$$

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